Time-nonlocal evolution equations

by

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Contents

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Declarations

The main results of this thesis are contained in Chapter 2, Chapter 3 and Chapter 4. They rearrange the content of the articles Hernández-Hernández et al. [2017], Du et al. [2018] and Toniazzi [2019], respectively. The first article is joint work Elena M. Hernández-Hernández (University of Warwick, UK) and Vassili N. Kolokoltsov (University of Warwick, UK), meanwhile the second article is joint work with Qiang Du (Columbia University, NY) and Zhi Zhou (Hong Kong Polytechnic University, HK).

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Abstract

Time-nonlocal evolution equations (EEs) are popular theoretical and applied models that introduce time-heterogeneity and memory effects in the classical heat equation. In this thesis we focus on wellposedness and Feynman-Kac formulae for several time-nonlocal EEs. In particular, we define and study a time-nonlocal initial condition, endowing it with probabilistic intuition. The main contribution is the stochastic-classical-wellposedness for the Marchaud EE and the stochastic-weak-wellposedness for the Marchaud-type EE. In the latter case, the kernel defining the nonlocal time derivative is allowed to be time-dependent. As a particular case we treat the Caputo-type inhomogeneous EE.

Chapter 1

Introduction and preliminaries

1.1 Introduction

Time-nonlocal evolution equations (EEs) extend the standard heat equation $\partial_t u = \Delta u$ by substituting the time derivative ∂_t with natural choices of nonlocal derivatives. A popular choice is the Caputo fractional derivative of order $\beta \in (0,1)$, denoted by D_0^β, leading to a model for sub-diffusion phenomena, with applications in a variety of fields such as physics, finance, biology and geology. We refer to [Meerschaert and Sikorskii, 2012, Chapter 2.4] for an overview. Probabilistically, the fundamental solution to the Caputo time-fractional EE $D_0^\beta u = \Delta u$ is the law of a time-changed Brownian motion $Y(t) = B(\tau_0^\beta(t))$, where τ_0^β is the inverse of a β-stable subordinator independent of B, which is a Brownian motion. The non-Markovian process Y has recently attracted a lot of attention, partly because it is a sub-diffusive process with rather surprising universality properties Barlow and Černý [2011]; Hairer et al. [2018]. Moreover, in the last two decades, generalisations of Caputo time-fractional EEs are becoming quite popular. Some motivations are their interesting properties, a rich parametrizability, theoretical and numerical tractability, see, e.g., Meerschaert et al. [2011]; Kochubei [2011]; Meerschaert and Scheffler [2006, 2005]; Chen [2017].

This thesis mainly concerns wellposedness and stochastic representation for the solutions of time-nonlocal EEs of the kind

$$\begin{cases} D_\infty^{(\nu)} u(t,x) = \mathcal{L}_\Omega u(t,x) + g(t,x), & \text{on } (0,T] \times \Omega, \\ \quad u(t,x) = \phi(t,x), & \text{on } (-\delta, 0] \times \Omega, \\ \quad u(t,x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$, the spatial operator $\mathcal{L}_\Omega$ is the generator of a Ω-valued sub-Markov process X^Ω, the horizon parameter $\delta \in (0,\infty]$ is the length of the support of the time dependent Lévy kernel ν and

$$-D_\infty^{(\nu)} u(t) = \int_0^\infty (u(t-r) - u(t))\, \nu(t,r) dr$$

is the generator of a *decreasing* $(-\infty, T]$-valued Markov process $-X^{(\nu)}$. Note that the EE (1.1) features time-nonlocal initial conditions. Such initial conditions have been recently introduced in Du et al. [2017]; Chen et al. [2017]. We prove that the solution allows the stochastic representation/Feynman-Kac formula

$$u(t,x) = \mathbf{E}\left[\phi\left(-X^{t,(\nu)}(\tau_0(t)), X^{x,\Omega}(\tau_0(t))\right)\right] + \mathbf{E}\left[\int_0^{\tau_0(t)} g\left(-X^{t,(\nu)}(s), X^{x,\Omega}(s)\right)\, ds\right], \quad (1.2)$$

where t and x refer to the starting points of the independent processes $-X^{(\nu)}$ and X^{Ω}, respectively, and $\tau_0(t)$ is the first exit time of $-X^{t,(\nu)}$ from $(0,T]$. We call problem (1.1) the Marchaud-type EE. The Marchaud-type EE is very general. To see this, let $\phi(t)=\phi(0)$ for all $t<0$, then one obtains the homogeneous Caputo-type EE and its solution Chen [2017]; Hernández-Hernández et al. [2017], which imposes initial conditions on $\{0\}\times\Omega$, as it is usual for EEs. If furthermore $\nu(t,r)=\nu(r)=\beta r^{-1-\beta}/\Gamma(1-\beta)$, then one obtains the standard Caputo EE and its well-known solution Meerschaert and Sikorskii [2012]. We summarise in the next pseudo-theorem the two main results of this thesis.

Theorem 1.1.1. *Suppose $-D_{\infty}^{(\nu)}$ generates a decreasing Markov process $-X^{(\nu)}$ such that $\mathbf{E}[\tau_0(T)]$ is finite, and that the spatial operator $\mathcal{L}_{\Omega}$ is the generator of an independent sub-Markov process $X^{x,\Omega}$. Also assume that both $-X^{t,(\nu)}(s)$ and $X^{x,\Omega}(s)$ allow a density for each $t,s>0$, $x\in\Omega$.*

(A) (Theorem 2.1.4)[1] Let $g\in L^{\infty}((0,T)\times\Omega)$ and $\phi(0)\in Dom(\mathcal{L}_{\Omega})$. Then there exists a unique generalised solution[2] to the Caputo-type EE (1.1), and the generalised solution allows the stochastic representation (1.2).
(Theorem 3.2.10) Furthermore, if $\mathcal{L}_{\Omega}=\Delta$ for a regular domain Ω, $g\in L^{\infty}(0,T;H_0^1(\Omega))$ and $\phi\in L^{\infty}((-\delta,0);H_0^1(\Omega))$, then the stochastic representation (1.2) is a weak solution to the Marchaud-type EE (1.1). For certain Lévy kernels $\nu(t,r)=p(t)q(r)$ the weak solution is unique.

(B) (Theorem 4.3.6) Consider the fractional stable-kernel $\nu(t,r)=\beta r^{-1-\beta}/\Gamma(1-\beta)$ and let $\mathcal{L}_{\Omega}=-(-\Delta)^{\alpha}|_{\Omega}$ be the restricted fractional Laplacian, for $\beta,\alpha\in(0,1)$ and a regular domain $\Omega\subset\mathbb{R}^d$. Let $g\in C^{1,2k}_{\partial\Omega}([0,T]\times\Omega)$ and $\phi\in C^{2,2k}_{b,\partial\Omega}((-\infty,0]\times\Omega)$, for large enough k depending on d and α. Then there exists a unique classical solution to the EE (1.1), and the classical solution allows the stochastic representation (1.2).

1.1.1 Main contribution

To the best of our knowledge of the existing literature, the main contributions of Theorem 1.1.1 can be summarised as follows.

(i) We construct a very general theory for a unified treatment of stochastic weak solutions to time-nonlocal[3] EEs of the kind (1.1), featuring time nonlocal initial conditions. Kernels ν with t-dependence are untreated in the literature[4], also for Caputo-type EEs (ϕ constant

[1]The notation for the function spaces can be found in Section 1.2.
[2]Generalised solutions are defined as the pointwise limit of solutions to abstract evolution equations obtained through semigroup theory.
[3]Our theory treats to some extent also local/drift type of time-operators.
[4]For for time-independent kernels $\nu(t,r)=\nu(r)$: the works Du et al. [2017]; Chen et al. [2017] treat weak solutions for EEs for specific spatial operators $\mathcal{L}_{\Omega}$, meanwhile the works Chen [2017]; Chen et al. [2018] treat stochastic strong solutions for homogeneous Caputo-type EEs.

in time). As special cases we treat Caputo EEs and Caputo-type EEs[5].

(ii) Our probabilistic approach for weak solutions appears to be new (at least in the fractional calculus literature), by treating the EE (1.1) as an elliptic boundary value problem. This shows why the non-Markovian time-change τ_0 in the solution (1.2) arises naturally and in great generality. Moreover, our approach requires minimal properties for time-operators and does not invoke the Laplace transform of Caputo-type operators[6]

(iii) The stochastic representation/solution (1.2) is new for problem (1.1). The inhomogeneous term in (1.2) is even new for the standard Caputo EE. Formula (1.2) (for $g = 0$) is particularly interesting because it allows to impose dependency of the initial condition ϕ on the waiting/trapping time of the anomalous diffusion $Y(t) = X^{\Omega}(\tau_0(t))$, which is indeed $W(t) = X^{t,(\nu)}(\tau_0(t))$. This shows that the time-nonlocal initial condition is not only natural theoretically, but also enriches the probabilistic intuition of the standard Caputo-type EEs.

(iv) We prove the first result about classical solutions for the space-time fractional version of the EE (1.1) in Theorem 4.3.6. We also prove existence and uniqueness for classical solutions[7] for the space-fractional inhomogeneous Caputo EE in Theorem 4.2.6.

1.1.2 Thesis structure

In the current chapter we introduce some general notation, basics on nonlocal and fractional calculus, and lastly a technical result for Feller semigroups. Chapter 2, Chapter 3 and Chapter 4 are the core of the thesis[8]. A brief summary of the welposedness results in such chapters is given in Section 1.5. We remark that Theorem 2.1.4 in Chapter 2 is essential for the weak solutions in Chapter 3 and Chapter 4, but otherwise these three chapters are independent. Finally, Chapter 5 provides some numerics to support our theoretical results and a discussion of the intuition for time-nonlocal initial conditions.

1.2 General notation, function spaces and Feller semigroups

We denote by $\mathbb{N}$, $\mathbb{R}^d$, $\mathbf{1}_A(\cdot)$, $a \wedge b$, a.e., a.s., lhs and rhs, the set of natural numbers, the d-dimensional Euclidean space, the indicator function of the set A, the minimum between $a, b \in \mathbb{R}$, the statements almost everywhere with respect to Lebesgue measure, almost surely, left hand side and right hand side, respectively. We define the one parameter Mittag-Leffler function for

[5]Again, for t-independent kernels ν, a general work on homogeneous Caputo-type EEs is given by Chen [2017]; Chen et al. [2018], proving strong solutions assuming $\int_0^\infty \nu(r)\,dr = \infty$, under the weaker assumption that Ω is a locally compact measure metric space. Other works treating specific Caputo-type EEs are Meerschaert and Scheffler [2006]; Meerschaert et al. [2011].

[6]The Laplace transform of Caputo-type operators seems essential for uniqueness arguments in all the probabilistic literature on time-fractional/nonlocal EEs.

[7]Existence of classical solutions of the Caputo inhomogeneous EE is obtained in Theorem Eidelman and Kochubei [2004], for uniformly elliptic second order operators and in Allen et al. [2016] for nonlocal spatial operators with symmetric kernels, both works for $\Omega = \mathbb{R}^d$. As far as we know, the inhomogeneous part of the stochastic representation (1.2) is new, and this our main contribution in terms of classical wellposedness of Caputo evolution equations.

[8]Each chapter is a rearrangement of the articles Hernández-Hernández et al. [2017], Du et al. [2018] and Toniazzi [2019], respectively.

$\beta \in (0,1)$ as $E_\beta(t) = \sum_{k=0}^{\infty} t^k \Gamma(k\beta+1)^{-1}$, $t \geq 0$, where $\Gamma(\lambda) := \int_0^\infty s^{\lambda-1} e^{-s}\, ds$ is the gamma function. We denote by $\|L\|$ the operator norm of a bounded linear operator L between Banach spaces. We generally denote by $\|x\|_B$ the norm of $x \in B$, for B a Banach space. To ease notation, $F(I) = FI$ whenever $F(I)$ is a space of real-valued functions on an interval $I \subset \mathbb{R}$. We define $C(A) = \{f : A \to \mathbb{R} \text{ is continuous}\}$, where A is any subset of $\mathbb{R}^d$. We define the Banach spaces

$$\begin{aligned}
B(A) &= \{f : A \to \mathbb{R} \text{ is bounded and measurable}\},\\
C_\infty(A) &= \{f \in C(A) \text{ and vanishes at infinity}\},\\
C(K) &= C_\infty(K),\\
C_{\partial\Omega}(\Omega) &= \{f \in C(\overline{\Omega}) : f = 0 \text{ on } \partial\Omega\},\\
C_0[0,T] &= \{f \in C[0,T] : f(0) = 0\},\\
C_{\partial\Omega}([0,T]\times\Omega) &= \{f \in C([0,T]\times\overline{\Omega}) : f = 0 \text{ on } \partial\Omega\},\\
C_{0,\partial\Omega}([0,T]\times\Omega) &= \{f \in C_{\partial\Omega}([0,T]\times\Omega) : f(0) = 0\},\\
C_{\infty,\partial\Omega}((-\infty,T]\times\Omega) &= \{f \in C_\infty((-\infty,T]\times\overline{\Omega}) : f = 0 \text{ on } \partial\Omega\},\\
C_{b,\partial\Omega}((-\infty,T]\times\Omega) &= \{f \in C((-\infty,T]\times\overline{\Omega}) : f \text{ is bounded and } f = 0 \text{ on } \partial\Omega\},
\end{aligned}$$

all equipped with the supremum norm $\|\cdot\|_\infty$, where the set $K \subset \mathbb{R}^d$ is compact, the set $\Omega \subset \mathbb{R}^d$ is bounded and open, $T \geq 0$, and we say that $f : A \to \mathbb{R}$ *vanishes at infinity* if given $\varepsilon > 0$, there exists $K \subset A$ compact such that $|f| \leq \varepsilon$ on $A\backslash K$. If $\Omega = \mathbb{R}^d$, we write

$$C_\infty(\Omega) = C_{\partial\Omega}(\Omega).$$

We define the spaces

$$\begin{aligned}
C^k(\Omega) &= \{f \in C(\Omega) : f \text{ is } k\text{-times continuously differentiable}\},\\
C_c^k(\Omega) &= \{f \in C(\Omega) : f \in C^k(\Omega) \text{ and compactly supported}\},\\
C_c^\infty(\Omega) &= \{f \in C(\Omega) : f \text{ is smooth and compactly supported}\},\\
C^1[0,T] &= \{f, f' \in C[0,T]\},\\
C_0^1[0,T] &= \{f, f' \in C_0[0,T]\},\\
C_\infty^1(-\infty,T] &= \{f, f' \in C_\infty(-\infty,T]\},\\
C^{1,k}((0,T)\times\Omega) &= \{f \in C((0,T)\times\Omega) : f \text{ is 1-time and } k\text{-times continuously}\\
&\qquad \text{differentiable in time and space, respectively}\},\\
C_c^{1,k}((0,T)\times\Omega) &= \{f \in C^{1,k}((0,T)\times\Omega) : f \text{ is compactly supported}\},\\
C_{\partial\Omega}^1([0,T]\times\Omega) &= \{f \in C_{\partial\Omega}([0,T]\times\Omega) : f \in C^{1,0}((0,T)\times\Omega), f' \in C_{\partial\Omega}([0,T]\times\Omega)\},\\
C_{b,\partial\Omega}^1((-\infty,T]\times\Omega) &= \{f, \partial_t f \in C_{b,\partial\Omega}((-\infty,T]\times\Omega)\},\\
C_{\infty,\partial\Omega}^{n,k}((-\infty,T]\times\Omega) &= \{f \in C_{\infty,\partial\Omega}((-\infty,T]\times\Omega) : \text{all derivatives up to order } n \text{ in time}\\
&\qquad \text{and } k \text{ in space exist and belong to } C_{\infty,\partial\Omega}((-\infty,T]\times\Omega)\},
\end{aligned}$$

where the set $O \subset \mathbb{R}^d$ is open and $n, k \in \mathbb{N}$. We write $C^{1,0}_{\infty,\partial\Omega}((-\infty,T]\times\Omega) = C^{1}_{\infty,\partial\Omega}((-\infty,T]\times\Omega)$. By $L^p(O)$ we mean the standard Banach spaces of real-valued Lebesgue p-integrable functions on O, $p \in [0,\infty]$. Without risk of confusion we might write $\|\cdot\|_{L^\infty(O)} = \|\cdot\|_\infty$. We denote by $W^{1,p}(\Omega)$ the standard Sobolev space of p-integrable functions on Ω with p-integrable weak first derivatives, $p \in [1,\infty]$. Denote by $H^{-1}(\Omega)$ the dual of $H^1_0(\Omega)$, where $H^1_0(\Omega)$ is the closure of $C^\infty_c(\Omega)$ in $W^{1,2}(\Omega)$. We define by $L^p(a,b;B) = \{f : (a,b) \to B \text{ such that } t \mapsto \|f(t)\|_B \in L^p(a,b)\}$, for $p \in [1,\infty]$ and $b > a \geq -\infty$.
Given two sets of real-valued functions F and $\tilde{F}$, we define $F \cdot \tilde{F} := \{f\tilde{f} : f \in F,\ \tilde{f} \in \tilde{F}\}$, and by $\mathrm{Span}\{F\}$ we mean the set of all linear combinations of functions in F.

The notation we use for an E-valued stochastic process started at $x \in E$ is $X^x = \{X^x(s)\}_{s\geq 0}$. Note that the symbol t will often be used to denote the starting point of a stochastic process with state space $E \subset \mathbb{R}$. We use the standard notation $\mathbf{E}$ and $\mathbf{P}$ for the mathematical expectation and probability, respectively. By a *strongly continuous contraction semigroup* P we mean a collection of linear operators $P_s : B \to B$, $s \geq 0$, where B is a Banach space, such that $P_{s+r} = P_sP_r$, for every $s, r \geq 0$, P_0 is the identity operator, $\lim_{s\downarrow 0} P_s f = f$ in B, for every $f \in B$, and $\sup_s \|P_s\| \leq 1$. The generator of the semigroup P is defined as the pair $(\mathcal{L}, \mathrm{Dom}(\mathcal{L}))$, where $\mathrm{Dom}(\mathcal{L}) := \{f \in B : \mathcal{L}f := \lim_{s\downarrow 0} s^{-1}(P_sf - f) \text{ exists in } B\}$. We say that a set $C \subset \mathrm{Dom}(\mathcal{L})$ is a *core for* $(\mathcal{L}, \mathrm{Dom}(\mathcal{L}))$ if the generator equals the closure of the restriction of $\mathcal{L}$ to C. We say that a set $C \subset B$ is *invariant under* P if $P_sC \subset C$ for every $s \geq 0$. If a set C is invariant under P and a core for $(\mathcal{L}, \mathrm{Dom}(\mathcal{L}))$, then we say that C is an *invariant core for* $(\mathcal{L}, \mathrm{Dom}(\mathcal{L}))$. For a given $\lambda \geq 0$ we define the *resolvent of* P by $(\lambda - \mathcal{L})^{-1} := \int_0^\infty e^{-\lambda s} P_s\, ds$. For $\lambda = 0$ we might call $(-\mathcal{L})^{-1}$ the *potential of* P. Recall that for $\lambda > 0$, $(\lambda - \mathcal{L})^{-1} : B \to \mathrm{Dom}(\mathcal{L})$ is a bijection and it solves the abstract resolvent equation

$$\mathcal{L}(\lambda - \mathcal{L})^{-1}f = \lambda(\lambda - \mathcal{L})^{-1}f - f, \quad f \in B,$$

see for example [Dynkin, 1965, Theorem 1.1]. We recall the Hille-Yosida Theorem stating that $(\mathcal{L}, \mathrm{Dom}(\mathcal{L}))$ is the generator of a strongly continuous contraction semigroup on B if and only if it is a closed operator, $(\lambda - \mathcal{L})\mathrm{Dom}(\mathcal{L})$ is dense in B for some $\lambda > 0$, and $\mathcal{L}$ is dissipative [Ethier and Kurtz, 2009, Chapter 1, Theorem 2.6]. By a *Feller semigroup* we mean a strongly continuous contraction semigroup P on any of the (compactified) Banach spaces of continuous functions defined above such that P preserves non-negative functions, and we call any such triplet $(P, B, (\mathcal{L}, \mathrm{Dom}(\mathcal{L})))$ a *Feller triplet*. A Feller semigroup P is said to be *conservative* if the extension of P to bounded measurable functions preserves constants. Feller semigroups are in one-to-one correspondence with Feller processes, where a Feller process is a time-homogenous Markov process $\{X(s)\}_{s\geq 0}$ such that $s \mapsto P_sf(x) := \mathbf{E}[f(X(s))|X(0) = x]$, $f \in B$ is a Feller semigroup [Böttcher et al., 2013, Chapter 1.2]. We recall that every Feller process admits a cádlág modification which enjoys the strong Markov property [Böttcher et al., 2013, Theorem 1.19 and Theorem 1.20], and we always work with such modification.

1.3 Nonlocal calculus

In this section we define the main generalised time-nonlocal derivatives used in this work. We provide some assumptions on the kernel ν, along with relevant probabilistic properties of such derivatives. The reader unfamiliar with the nonlocal operators here presented, might find it preferable to read first Section 1.4, Section 1.6 and then Chapter 4, in order to first cover the fractional counterpart of this thesis.

1.3.1 Main assumptions

For convenience, we state here the main assumptions that will be used in this thesis. The notation for assumptions (H1a), (H1a') and (H1b) is defined in Section 1.3.3.

(H0) Let $\nu : \mathbb{R} \times (0,\infty) \to [0,\infty)$ be a non-negative continuous function which is continuously differentiable in the first variable. Furthermore, we assume the first moment uniform bounds

$$\sup_t \int_0^\infty (1 \wedge r)\nu(t,r)\, dr < \infty, \quad \sup_t \int_0^\infty (1 \wedge r)\Big|\frac{\partial}{\partial t}\nu(t,r)\Big|\, dr < \infty,$$

the tightness condition

$$\lim_{\delta \to 0} \sup_t \int_{0<r\le\delta} r\nu(t,r)\, dr = 0,$$

and the ellipticity condition given by the existence of $\epsilon, \gamma > 0$ such that $\nu(t,r) \ge \gamma > 0$ for all $t \in \mathbb{R}$ and $|r| < \epsilon$.

(H0') Assumption (H0) holds, and also

$$\int_0^\infty (1 \wedge r) \sup_t \nu(t,r)\, dr < \infty, \quad \int_0^\infty (1 \wedge r) \sup_t \Big|\frac{\partial}{\partial t}\nu(t,r)\Big|\, dr < \infty.$$

(H1a) The law of $-X^{t,(\nu)}(s)$ is absolutely continuous with respect to Lebesgue measure for each $t \in [0,T]$, $s > 0$, and we denote such density by $p_s^{(\nu)}(t)$.

(H1a') Assumption (H1a) holds and $\mathbf{P}[-X^{t,(\nu)}(\tau_0(t)) \in \{0\}] = 0$, for each $t \in (0,T]$.

(H1b) The law of $X^{x,\Omega}(s)$ is absolutely continuous with respect to Lebesgue measure for each $x \in \Omega$, $s > 0$, and we denote such density by $p_s^{\Omega}(x)$.

(H2) The kernel ν satisfies $\nu(t,r) \ge Cr^{-1-\beta}$ for all $t \in \mathbb{R}$, for some constant $C > 0$ and $\beta \in (0,1)$.

Remark 1.3.1. Assumption (H1a) helps us handle the weak problem data (see Theorem 2.1.4-(ii)). Otherwise, without (H1a), we could assume that the problem data g in Theorem 2.1.4-(ii) is a Baire class 1 function (see Remark 2.1.7). This would allow us to handle several cases, such as ν being integrable, for our notion of generalised solution.

Remark 1.3.2. Assumption $\mathbf{P}[-X^{t,(\nu)}(\tau_0(t)) \in \{0\}] = 0$ is implied by the (H1a) if $\nu(t,r)dr = \nu(dr)$. This is because the existence of a density implies that $\nu((0,\infty)) = \infty$, as $X^{(\nu)}$ cannot be a compound Poisson process [Sato, 1999, Remark 27.3]. Then $\tau_0(t) = \inf\{s > 0 : X^{(\nu)}(s) > t\}$, the right inverse of $X^{(\nu)}$, and one can apply [Bertoin, 1996, III, Theorem 4]. Here $X^{(\nu)}$ is the increasing subordinator with Lévy measure $\nu(dr)$.

Example 1.3.3. Some examples where the densities $p_s^{(\nu)}(t)$, $t, s > 0$ exist are given by:

(i) kernels $\nu(t, r)dr = \nu(dr)$ and $\nu(dr) \geq r^{-1-\alpha}dr$ for all small r [Sato, 1999, Proposition 28.3]. Combine with Remark 1.3.2, (H2) implies (H1a') for translation invariant kernels;

(ii) kernels $\nu(t, r) = \nu(r)$, $\int_0^\infty \nu(r)\, dr = \infty$ [Sato, 1999, Theorem 27.7];

(iii) kernels $\nu(t, r)$ such that the respective symbols satisfies the Hölder continuity-type conditions in [Kühn, 2017, Theorem 2.14];

(iv) see Fournier [2002] for another set of assumptions for kernels of the type $\nu(t, r) = p(t)q(r)$ and a literature discussion.

1.3.2 Caputo-type and Marchaud-type derivatives

Definition 1.3.4. For any kernel function ν satisfying condition (H0) and $T > 0$, the *Marchaud-type derivative* $D_\infty^{(\nu)}$ is defined by

$$D_\infty^{(\nu)} u(t) = \int_0^\infty (u(t) - u(t - r))\nu(t, r)\, dr, \quad t \in (-\infty, T], \tag{1.3}$$

and the *Caputo-type derivative* $D_0^{(\nu)}$ is defined by

$$D_0^{(\nu)} u(t) = \int_0^t (u(t) - u(t - r))\nu(t, r)\, dr + (u(t) - u(0)) \int_t^\infty \nu(t, r)\, dr, \quad t \in (0, T], \tag{1.4}$$

and $D_0^{(\nu)} u(0) = \lim_{t\downarrow 0} D_0^{(\nu)} u(t)$.

Remark 1.3.5. Note that $D_\infty^{(\nu)}$ and $D_0^{(\nu)}$ are welldefined on $C_b^1(-\infty, T]$ and $C^1[0, T]$, respectively, and $D_\infty^{(\nu)} = D_0^{(\nu)}$ on $C_b^1(-\infty, T] \cap \{f(t) = f(0) \text{ for } t < 0\}$. Also note that both $-D_\infty^{(\nu)}$ and $-D_\infty^{(\nu)}$ satisfy the positive maximum principle[9].

Remark 1.3.6. The nonlocal derivative $D_\infty^{(\nu)}$ can be seen as the left-sided generalisation of the *Marchaud derivative* [Samko and Marichev, 1993, eq. (5.57) and (5.58)]. This operator is also known as the *generator form of fractional derivatives* Kolokoltsov [2011]; Meerschaert and Sikorskii [2012], or a Lévy-type generator Böttcher et al. [2013].

Remark 1.3.7. The nonlocal derivatives $-D_\infty^{(\nu)}$ and $-D_0^{(\nu)}$ have a clear probabilistic interpretation. The former tells us that the process at t makes a negative jump of size $|r|$ with intensity $\nu(t, r)$. The latter tells us that, as long as the jump does not cross 0, the process jumps from t to $t - r$ with intensity $\nu(t, r)$. Otherwise, it gets killed with rate/intensity $\int_t^\infty \nu(t, r)\, dr$ and regenerated at 0 with the same rate, where it remains absorbed. This will be made rigorous in Proposition 1.3.11

[9] I.e. $-D_\infty^{(\nu)} f(t^*) \leq 0$ and $-D_0^{(\nu)} f(t^*) \leq 0$ if $f(t^*) = \max_t f(t) > 0$, for appropriate f's.

Remark 1.3.8. The previous remark suggests that the initial value problem for $-D_{\infty}^{(\nu)}$ should require boundary conditions on $(-\infty, 0]$, i.e.

$$-D_{\infty}^{(\nu)}u = -f, \text{ on } (0,T], \quad u = \phi \text{ on } (-\infty, 0],$$

meanwhile initial value problem for $-D_0^{(\nu)}$ should require boundary conditions on $\{0\}$, i.e.

$$-D_0^{(\nu)}u = -f, \text{ on } (0,T], \quad u = \phi \text{ on } \{0\},$$

and their solutions can be easily guessed probabilistically. We will extend this viewpoint to EEs such as $-D_{\infty}^{(\nu)}u + \Delta u = 0$ and $-D_0^{(\nu)}u + \Delta u = 0$.

Example 1.3.9. We mention some concrete and popular examples of the nonlocal operators.

(i) By setting $\nu(t,r) = -r^{-\beta-1}/\Gamma(-\beta)$ with $\beta \in (0,1)$, the nonlocal operator $D_0^{(\nu)}$ reproduces the Caputo fractional derivative and $D_{\infty}^{(\nu)}$ the Marchaud fractional derivative, which we define in Section 1.4.1.

(ii) Tempered Lévy kernels are obtained if $\nu(t,r) = -e^{-\lambda r}r^{-1-\beta}/\Gamma(-\beta)$, $\beta \in (0,1)$, $\lambda > 0$ Chakrabarty and Meerschaert [2011]; Wylomańska [2013].

(iii) Fractional derivatives of variable order are obtained by taking $\nu(t,r) = -r^{-1-\beta(t)}/\Gamma(-\beta(t))$ with a suitable function $\beta(t) : \mathbb{R} \to (0,1)$ Hernández-Hernández and Kolokoltsov [2016].

(iv) The operator $\mathcal{G}_\delta$, defined in [Du et al., 2017, formula (1.2)], is a special case of the Marchaud-type operator $D_{\infty}^{(\nu)}$ with a time-independent and compactly supported kernel function.

1.3.3 Related time-valued Feller semigroups

In this section, we discuss three stochastic processes generated by the derivatives defined in (1.3) and (1.4) with kernel functions satisfying (H0).

Definition 1.3.10. Assume (H0).

(i) [Kolokoltsov, 2011, Theorem 5.1.1]: Let $(P^{(\nu),\infty}, C_\infty(-\infty,T], (\mathcal{L}_{(\nu)}^{\infty}, \mathrm{Dom}(\mathcal{L}_{(\nu)}^{\infty})))$ be the Feller triplet where

$$\left(\mathcal{L}_{(\nu)}^{\infty}, \mathrm{Dom}(\mathcal{L}_{(\nu)}^{\infty})\right) \text{ is the closure of } \left(-D_{\infty}^{(\nu)}, C_{\infty}^{1}(-\infty,T]\right),$$

and recall that $C_{\infty}^{1}(-\infty,T]$ is invariant under $P^{(\nu),\infty}$. Denote the induced Feller process when started at $t \in (-\infty,T]$ by

$$-X^{t,(\nu)} = \{-X^{t,(\nu)}(s)\}_{s\geq 0}.$$

(ii) Proposition 1.3.11: Let $(P^{(\nu)}, C[0,T], (\mathcal{L}_{(\nu)}, \mathrm{Dom}(\mathcal{L}_{(\nu)}))$ be the Feller triplet where

$$\left(\mathcal{L}_{(\nu)}, \mathrm{Dom}(\mathcal{L}_{(\nu)})\right) \text{ is the closure of } \left(-D_0^{(\nu)}, C^1[0,T] \cap \{g'(0) = 0\}\right),$$

and recall that $C^1[0,T]\cap\{g'(0)=0\}$ is invariant under $P^{(\nu)}$. Denote the induced Feller process when started at $t\in[0,T]$ by

$$-X_0^{t,(\nu)}=\{-X^{t,(\nu)}(s)\mathbf{1}_{\{s<\tau_0(t)\}}\}_{s\geq 0}.$$

(iii) Proposition 1.3.11: Let $(P^{(\nu),\text{kill}},C_0[0,T],(\mathcal{L}^{\text{kill}}_{(\nu)},\text{Dom}(\mathcal{L}^{\text{kill}}_{(\nu)}))$ be the Feller triplet where

$$\left(\mathcal{L}^{\text{kill}}_{(\nu)},\text{Dom}(\mathcal{L}^{\text{kill}}_{(\nu)})\right)\ \text{ is the closure of }\ \left(-D_0^{(\nu)},C_0^1[0,T]\right),$$

and recall that $C_0^1[0,T]$ is invariant under $P^{(\nu),\text{kill}}$.

The next proposition justifies the use of $C_\infty(-\infty,T]$ instead of $C_\infty(\mathbb{R})$ in Definition 1.3.10-(i). We remark that in the articles Du et al. [2018]; Hernández-Hernández et al. [2017] we use [Kolokoltsov, 2015, Theorem 4.1], but the next proposition allows us to bypass such theorem, obtaining the desired semigroups and invariant cores directly from the process in Definition 1.3.10-(i).

Proposition 1.3.11. (i) The processes $-X^{t,(\nu)}$ is non-increasing.

(ii) The expectation of the first exit times

$$\tau_0(t):=\inf\{s>0:-X^{t,(\nu)}(s)\leq 0\},\quad t\in(0,T],$$

is uniformly bounded, in the sense that $\sup_{t\in(0,T]}\mathbf{E}[\tau_0(t)]<\infty$.

(iii) The absorbed process $-X_0^{(\nu)}$ defined by

$$-X_0^{t,(\nu)}(s):=\begin{cases}-X^{t,(\nu)}(s), & \text{if } s<\tau_0(t),\\ 0, & \text{if } s\geq\tau_0(t),\end{cases}$$

is a Feller process and its respective Feller triplet is the one in Definition 1.3.10-(ii). The killed version of $-X_0^{t,(\nu)}$ is also a Feller process, and its respective Feller triplet is the one in Definition 1.3.10-(iii). It also holds that $P^{(\nu)}=P^{(\nu),\text{kill}}$ on $C_0[0,T]$.

(iv) It holds that $(\mathcal{L}^{\text{kill}}_{(\nu)},\text{Dom}(\mathcal{L}^{\text{kill}}_{(\nu)}))=(\mathcal{L}_{(\nu)},\text{Dom}(\mathcal{L}_{(\nu)})\cap\{f(0)=0\})$.

Proof.

(i) This is a simplified version of the proof of [Hernández-Hernández et al., 2017, Proposition 2.7-(i)] and we omit it.

(ii) This follows for example by the comparison principle in [Zhang, 2007, Theorem 1.5], as it proves for each $t,s>0$ the first inequality in

$$\mathbf{P}[-X^{t,(\nu)}(s)>0]\leq\mathbf{P}[t-X^{(\tilde{\nu})}(s)>0]\leq\mathbf{P}[T-X^{(\tilde{\nu})}(s)>0].$$

Here $X^{(\tilde{\nu})}$ is the non-decreasing compound Poisson process with Lévy kernel $\tilde{\nu}(r)=\gamma\mathbf{1}_{(0,\epsilon)}(r)$, where $\epsilon,\gamma>0$ are chosen as in (H0).

(iii) In this proof, for any $f \in C[0,T]$ we define the constant extension $\bar{f}(t) := f(t)\mathbf{1}_{[0,T]}(t) + f(0)\mathbf{1}_{(-\infty,0)}(t)$, $t \in (-\infty, T]$. We momentarily abuse notation defining for each $s \geq 0$

$$P_s^{(\nu)} f := P_s^{(\nu),\infty} \bar{f}, \quad f \in C[0,T].$$

It then follows by the Feller properties of $P^{(\nu),\infty}$ that $P^{(\nu)}$ is a Feller semigroup on $C[0,T]$ and on $C_0[0,T]$. We denote the latter semigroup by $P^{(\nu),\text{kill}}$. By part (i), it is clear that $P^{(\nu)}$ on $C[0,T]$ (on $C_0[0,T]$) is the Feller semigroup corresponding to $-X_0^{(\nu)}$ (the killed version of $-X_0^{(\nu)}$), resolving the ambiguity in the notation. Moreover

$$C_0^1[0,T] \subset \text{Dom}(\mathcal{L}_{(\nu)}^{\text{kill}}), \tag{1.5}$$

as $P^{(\nu),\infty}\bar{f} = P^{(\nu),\text{kill}} f$ and $f'(0+) = 0$ implies $\bar{f} \in C_\infty^1(-\infty, T]$, proving that $\mathcal{L}_{(\nu)}^{\text{kill}} f$ is welldefined and it equals $\mathcal{L}_{(\nu)}^{\infty}\bar{f}$ on $[0,T]$. Then

$$-D_0^{(\nu)} f = -D_\infty^{(\nu)} \bar{f} = \mathcal{L}_{(\nu)}^{\infty} \bar{f} = \mathcal{L}_{(\nu)}^{\text{kill}} f, \tag{1.6}$$

using [Kolokoltsov, 2011, Theorem 5.1.1] in the second equality. We claim that $C_0^1[0,1]$ is a core for $(\mathcal{L}_{(\nu)}^{\text{kill}}, \text{Dom}(\mathcal{L}_{(\nu)}^{\text{kill}}))$. First observe that

$$C_0^1[0,T] \text{ is invariant under } P^{(\nu),\text{kill}}. \tag{1.7}$$

This is because for each $f \in C_0^1[0,T]$, we have $\bar{f} \in C_\infty^1(-\infty, T]$, and we know by [Kolokoltsov, 2011, Theorem 5.1.1] that $C_\infty^1(-\infty, T]$ is invariant under $P^{(\nu),\infty}$. Then, for each $s > 0$, $P_s^{(\nu)} f \in C^1(0,T]$, and $\partial_t P_s^{(\nu),\infty} \bar{f}(0-) = 0$ implies $\partial_t P_s^{(\nu)} f(0+) = 0$. Combined with (1.5), $C_0^1[0,T]$ is a dense invariant subspace of the domain, and [Böttcher et al., 2013, Lemma 1.34] proves the claim (1.7).

Turning our attention to the absorbed process $-X_0^{(\nu)}$, let $f \in C^1[0,T] \cap \{g'(0+) = 0\}$, then by (1.5) and Part (iv)[10]

$$\tilde{f} := f - f(0) \in \text{Dom}(\mathcal{L}_{(\nu)}^{\text{kill}}) = \text{Dom}(\mathcal{L}_{(\nu)}) \cap \{g(0) = 0\}.$$

As $f(0)$ belongs to the linear space $\text{Dom}(\mathcal{L}_{(\nu)})$, it follows that $f \in \text{Dom}(\mathcal{L}_{(\nu)})$. Then

$$\mathcal{L}_{(\nu)} f = \mathcal{L}_{(\nu)}^{\text{kill}} \tilde{f} + \mathcal{L}_{(\nu)} f(0) = -D_0^{(\nu)} f,$$

using $\mathcal{L}_{(\nu)} f(0) = 0$ and (1.6) in the last equality. By [Böttcher et al., 2013, Lemma 1.34], it remains to show $C^1[0,T] \cap \{g'(0+) = 0\}$ is invariant under $P^{(\nu)}$. This is follows by

[10] This is not a circular argument as part (iv) of this Proposition is proved with an independent argument.

considering $f \in C^1[0,T] \cap \{g'(0)=0\}$ and computing for each $s>0$

$$\begin{aligned}
P_s^{(\nu)} f(t) &= P_s^{(\nu),\infty} \bar{f}(t) \\
&= P_s^{(\nu),\infty} (f \mathbf{1}_{\{\cdot>0\}})(t) + f(0) P_s^{(\nu),\infty} (\mathbf{1}_{\{\cdot\leq 0\}})(t) \\
&= P_s^{(\nu),\infty} ((f-f(0))\mathbf{1}_{\{\cdot>0\}})(t) + f(0) \\
&= P_s^{(\nu),\infty} \tilde{f}(t) + f(0) \\
&= P_s^{(\nu),\mathrm{kill}} \tilde{f}(t) + f(0) \in C^1[0,T] \cap \{g'(0)=0\}, \quad \text{for } t \geq 0,
\end{aligned}$$

using in the last line the fact that $\tilde{f} := \bar{f} - f(0) \in C_0^1[0,T]$ and (1.7).

(iv) This is the same argument as in Proposition 1.4.12-(ii) and we omit it.

□

Remark 1.3.12. If the Lévy kernel is independent of t, i.e. $\nu(t,r) = \nu(r)$, then $-X^{t,(\nu)}(s) = t - X^{(\nu)}(s)$ is the decreasing Lévy process with generator $-D_\infty^{(\nu)}$ acting on $C_c^\infty(\mathbb{R})$, where $X^{(\nu)}$ is the subordinator with Lévy measure $\nu(r)dr$. This is a consequence of the fact that $\mathcal{L}_{(\nu)}^\infty = -D_\infty^{(\nu)}$ on $C_c^\infty(\mathbb{R}) \subset \mathrm{Dom}(\mathcal{L}_{(\nu)}^\infty)$, and [Böttcher et al., 2013, Theorem 2.7].

Remark 1.3.13. Assumption (H0) could be replaced with an alternative one, as long as $-D_\infty^{(\nu)}$ generates a non-increasing Feller process such that Proposition 1.3.11-(ii) holds, along with the existence of an invariant core with the properties in Definition 1.3.10-(i). Nevertheless, assumption (H0) provides a satisfactory level of generality for most of the applications we have in mind.

1.3.4 The spatial operator $\mathcal{L}_\Omega$

We define in terms of semigroup theory a general class of spatial operators for our EEs.

Definition 1.3.14. For any $\Omega \subset \mathbb{R}^d$ open, we denote by

$$\left(P^\Omega, C_{\partial\Omega}(\Omega), (\mathcal{L}_\Omega, \mathrm{Dom}(\mathcal{L}_\Omega))\right)$$

any Feller triplet on $C_{\partial\Omega}(\Omega)$[11]. We denote the induced Feller process by

$$X^{x,\Omega} = \{X^{x,\Omega}(s)\}_{s\geq 0}$$

when started at $x \in \Omega$, and we define the first exit times for $x \in \Omega$

$$\tau_\Omega(x) := \inf\{s>0 : X^{x,\Omega}(s) \notin \Omega\}.$$

Remark 1.3.15. Most of our work assumes (H1b). There is a wide literature on this subject. We mention some examples with the exclusive intent of showing the variety of spatial operators that we can treat in Chapter 2, which includes several nonlocal and fractional derivatives on $\mathbb{R}^d$

[11] Recall $C_{\partial\Omega}(\Omega)$ is compactified with a cemetery state ∂ as usual [Böttcher et al., 2013, Chaper 1, Introduction]. Then, by the strong Markov property, the induced Feller process is absorbed at ∂ upon its first visit to ∂.

and on bounded domains with Dirichlet exterior conditions. Examples of Feller processes that satisfy (H1b) are

(i) All strong Feller Lévy processes ($\Omega = \mathbb{R}^d$). Indeed this is a characterisation [Hawkes, 1979, Lemma 2.1, p.338]. See [Kühn, 2017, Chapter 5.5] for a discussion. This class includes all stable Lévy processes.

(ii) See the conditions of the Lévy-type and Lévy measures in Example 1.3.3 and also [Kühn, 2017, Theorem 3.3].

(iii) Clearly any Feller processes X taking values in $\mathbb{R}^d$ such that its density is continuous. If then X is doubly Feller and $\Omega \subset \mathbb{R}$ is a bounded open regular set, then the process killed upon the first exit from Ω is a Feller process on Ω [Chung, 1986, p. 68], and it has a continuous density (which can be proved by the strong Markov property as in [Chen et al., 2012, formula (4.1)]).

(iv) We mention the articles Chen et al. [2010]; Grzywny and Szczypkowski [2018] and references therein for related discussions about some jump-type generators with symmetric and non-symmetric kernels.

Remark 1.3.16. To simplify the exposition, we only treat Dirichlet boundary conditions for bounded domains Ω.

1.4 Fractional calculus

We introduce four classical fractional derivatives that will be used throughout the thesis. The Caputo and Marchaud derivatives will be time derivatives in the EEs. The fractional Laplacian and the restricted fractional Laplacian will be spatial derivatives in the EEs.

1.4.1 Caputo and Marchaud derivatives

We refer to the books Samko and Marichev [1993] and Diethelm [2010] for an analytical introduction to Marchaud and Caputo derivatives, respectively. In this work we view Marchaud and Caputo derivatives as generators of decreasing Lévy processes, and we refer to Böttcher et al. [2013] and Bernyk et al. [2011] for a discussion of such probabilistic viewpoint.

Definition 1.4.1. For a parameter $\beta \in (0,1)$, we define the *Marchaud derivative* D_∞^β by

$$D_\infty^\beta u(t) = \int_0^\infty (u(t-r) - u(t)) \frac{\Gamma(-\beta)^{-1} dr}{r^{1+\beta}}, \quad t \in \mathbb{R},$$

and the *Caputo derivative* D_0^β by

$$D_0^\beta u(t) = \int_0^t (u(t-r) - u(t)) \frac{\Gamma(-\beta)^{-1} dr}{r^{1+\beta}} + (u(0) - u(t)) \int_t^\infty \frac{\Gamma(-\beta)^{-1} dr}{r^{1+\beta}}, \quad t > 0,$$

and $D_0^\beta u(0) = \lim_{t \downarrow 0} D_0^\beta u(t)$.

Remark 1.4.2. Note that Remark 1.3.5, Remark 1.3.7 and Remark 1.3.8 apply for D_0^β and D_∞^β, and if $f(0) = 0$, then D_0^β equals the Riemann-Liouville derivative (in generator form).

Proposition 1.4.3. The alternative representation of the Caputo derivative

$$D_0^\beta u(t) = \int_0^t u'(r)\frac{(t-r)^{-\beta}dr}{\Gamma(1-\beta)}, \quad \text{for } 0 < t < T,$$

holds if $u \in C[0,T] \cap C^1(0,T)$ and $u' \in L^1(0,T)$. Also, the Marchaud derivative equals

$$D_\infty^\beta u(t) = \int_{-\infty}^t u'(r)\frac{(t-r)^{-\beta}dr}{\Gamma(1-\beta)}, \quad \text{for } -\infty < t < T,$$

if $u \in C_\infty^1(-\infty,T]$ is such that $u' \in L^1(-\infty,0)$.

Proof. These are standard computations and can be found in [Kolokoltsov, 2015, Appendix]. □

1.4.2 Fractional Laplacians

We refer to Bogdan et al. [2009] for a probabilistic introduction to the fractional Laplacian and the restricted fractional Laplacian. We refer to Bonforte and Vázquez [2016]; Ros-Oton [2015]; Lischke et al. [2018]; Bucur [2015] for a more analytical introduction, related partial differential equations and applications.

Definition 1.4.4. For the parameter $\alpha \in (0,2)$ and $\Omega \subset \mathbb{R}^d$ open, we define the *restricted fractional Laplacian* $\Delta_\Omega^{\frac{\alpha}{2}}$ by

$$\Delta_\Omega^{\frac{\alpha}{2}} f(x) = \lim_{\varepsilon\downarrow 0}\int_{\Omega\setminus B_\varepsilon(x)} (f(y)-f(x))\frac{c_{\alpha,d}\,dy}{|x-y|^{d+\alpha}} - f(x)\int_{\mathbb{R}^d\setminus\Omega}\frac{c_{\alpha,d}\,dy}{|x-y|^{d+\alpha}}, \quad x \in \Omega,$$

and $\Delta_\Omega^{\frac{\alpha}{2}} f(z) = \lim_{x\to z}\Delta_\Omega^{\frac{\alpha}{2}} f(x)$ for $z \in \partial\Omega$, where $c_{\alpha,d}^{-1} = \int_{\mathbb{R}^d}\frac{1-\cos y_1}{|y|^{d+\alpha}}\,dy$, $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$ and $B_\varepsilon(x)$ denotes the Euclidean ball of radius $\varepsilon > 0$ around $x \in \Omega$. If $\Omega = \mathbb{R}^d$ then we write $\Delta_\Omega^{\frac{\alpha}{2}} = -(-\Delta^{\frac{\alpha}{2}})$, and we call the operator the *fractional Laplacian*.

Remark 1.4.5. Note that $\Delta_\Omega^{\frac{\alpha}{2}}$ is defined on functions on Ω, so that Euclidean boundary $\partial\Omega$ will make sense in the EE. Also note that $\Delta_\Omega^{\frac{\alpha}{2}}$ is welldefined on $C^2(\overline{\Omega})$ and it satisfies the positive maximum principle. See Bucur [2015]; Bonforte and Vázquez [2016] for concise discussions on representations of the restricted fractional Laplacian.

Remark 1.4.6. The probabilistic intuition for $\Delta_\Omega^{\frac{\alpha}{2}}$ is the following. The process at $x \in \Omega$ either jumps to $y \in \Omega$ with intensity $c_{\alpha,d}|x-y|^{-d-\alpha}$ or it gets killed when jumping outside of Ω, which happens with rate/intensity $\int_{\mathbb{R}^d\setminus\Omega}\frac{c_{\alpha,d}\,dy}{|x-y|^{d+\alpha}}$.

1.4.3 Fractional Feller semigroups

In this section we discuss properties of some Feller semigroups generated by the fractional derivatives introduced in Definition 1.4.1 and Definition 1.4.4.

Definition 1.4.7. For $\beta \in (0,1)$, we denote by $X^\beta = \{X^\beta(s)\}_{s\geq 0}$ the standard β-stable subordinator, characterised by the Laplace transforms $\mathbf{E}[e^{-kX^\beta(s)}] = e^{-k^\beta s}$, $k, s > 0$. We denote by p_s^β the smooth density of $X^\beta(s)$, $s > 0$.

Definition 1.4.8. For $\beta \in (0,1)$, we denote by

$$-X^{t,\beta} = \{-X^{t,\beta}(s) := t - X^\beta(s)\}_{s\geq 0}$$

the *inverted β-stable subordinator started at $t \in \mathbb{R}$*. We define the first exit/passage times $\tau_0^\beta(t) := \inf\{s > 0 : t - X^\beta(s) \leq 0\}$, $t \in \mathbb{R}$.

Definition 1.4.9. For $\alpha \in (0,2)$, $d \in \mathbb{N}$, we denote by $X^{x,\alpha} = \{X^{x,\alpha}(s)\}_{s\geq 0}$ the *rotationally symmetric α-stable Lévy process with values in $\mathbb{R}^d$, started at $x \in \mathbb{R}^d$*, with characteristic functions $\mathbf{E}[e^{ik\cdot X^{0,\alpha}(s)}] = e^{-s|k|^\alpha}$, $k \in \mathbb{R}^d$, $s > 0$. For any open set $\Omega \subset \mathbb{R}^d$, we define the first exit times $\tau_\Omega^\alpha(x) = \inf\{s > 0 : X^{x,\alpha}(s) \notin \Omega\}$, $x \in \mathbb{R}^d$.

Remark 1.4.10. In Chapter 4 it will always hold that $X^{t,(\nu)} = X^{t,\beta}$ and that $X^{x,\Omega} = X^{x,\alpha}$, hence we write $\tau_0^\beta = \tau_0$ and $\tau_\Omega^\alpha = \tau_\Omega$ to ease notation.

Remark 1.4.11. Recall that the smooth density of $-X^{t,\beta}(s)$, $s > 0$, is supported $(-\infty, t)$ and it equals $p_s^\beta(t-\cdot)$, and that the law of $X^{x,\alpha}(s)$ is smooth for each $s > 0$ (see for example [Bogdan et al., 2009, page 10]).

Proposition 1.4.12. Fix $T > 0$. For the the inverted β-stable subordinator $-X^{t,\beta}$, denote the Feller semigroup $P^{\beta,\infty} = \{P_s^{\beta,\infty}\}_{s\geq 0}$ on $C_\infty(-\infty, T]$, by $P_s^{\beta,\infty} f(t) := \mathbf{E}[f(-X^{t,\beta}(s))]$, $s \geq 0$, denote by $(\mathcal{L}_\beta^\infty, \mathrm{Dom}(\mathcal{L}_\beta^\infty))$ the generator of $P^{\beta,\infty}$, and recall that $C_\infty^1(-\infty, T]$ is an invariant core for $(\mathcal{L}_\beta^\infty, \mathrm{Dom}(\mathcal{L}_\beta^\infty))$ with $\mathcal{L}_\beta^\infty = -D_\infty^\beta$ on $C_\infty^1(-\infty, T]$.

(i) Define the absorbed process $-X_0^{t,\beta}$ by

$$-X_0^{t,\beta}(s) := \begin{cases} -X^{t,\beta}(s), & \text{if } s < \tau_0^\beta(t), \\ 0, & \text{if } s \geq \tau_0^\beta(t). \end{cases} \tag{1.8}$$

Then the process $-X_0^{t,\beta}$ induces a Feller semigroup on $C[0,T]$, denoted by $P^\beta = \{P_s^\beta\}_{s\geq 0}$, with generator $(\mathcal{L}_\beta, \mathrm{Dom}(\mathcal{L}_\beta))$. Moreover, $C^1[0,T]$ is an invariant core for $(\mathcal{L}_\beta, \mathrm{Dom}(\mathcal{L}_\beta))$ and

$$\mathcal{L}_\beta = -D_0^\beta \quad \text{on} \quad C^1[0,T].$$

(ii) The sub-Feller semigroup $P^{\beta,\mathrm{kill}} := P^\beta$ on $C_0[0,T]$ is the sub-Feller semigroup induced by the killed version of the process (1.8), and its generator is $(\mathcal{L}_\beta^{\mathrm{kill}}, \mathrm{Dom}(\mathcal{L}_\beta^{\mathrm{kill}})) = (\mathcal{L}_\beta, \mathrm{Dom}(\mathcal{L}_\beta) \cap \{f(0) = 0\})$. Moreover, $C_0^1[0,T]$ is an invariant core for $(\mathcal{L}_\beta^{\mathrm{kill}}, \mathrm{Dom}(\mathcal{L}_\beta^{\mathrm{kill}}))$ and

$$\mathcal{L}_\beta^{\mathrm{kill}} = -D_0^\beta \quad \text{on} \quad C_0^1[0,T].$$

(iii) The following three identities hold

$$\mathbf{E}\left[\tau_0^\beta(t)\right] = \frac{t^\beta}{\Gamma(\beta+1)}, \quad \mathbf{E}\left[e^{-\lambda\tau_0^\beta(t)}\right] = E_\beta(-\lambda t^\beta), \quad t, \lambda \geq 0, \text{ and} \tag{1.9}$$

$$\int_0^\infty p_s^\beta(t-r)\,ds = \frac{(t-r)^{\beta-1}}{\Gamma(\beta)}, \quad t>r. \tag{1.10}$$

Proof.

(i) It is easy to prove that $P_s^\beta f(t) := \int_0^t f(r)p_s^\beta(t-r)\,dr + f(0)\int_{-\infty}^0 p_s^\beta(t-r)\,dr$ is a Feller semigroup on $C[0,T]$, and the corresponding process is indeed $-X_0^{t,\beta}$. By using the proof of [Bernyk et al., 2011, Proposition 14][12], it holds that $C^1[0,T] \subset \mathrm{Dom}(\mathcal{L}_\beta)$, and that $\mathcal{L}_\beta = -D_0^\beta$ on $C^1([0,T])$. To prove that $C^1([0,T])$ is invariant under P^β, we directly compute for $g \in C^1([0,T])$, $t \in (0,T)$ and $s>0$,

$$\begin{aligned}\partial_t P_s^\beta g(t) &= \partial_t\left(\int_0^t g(t-r)p_s^\beta(r)\,dr + g(0)\int_{-\infty}^{-t} p_s^\beta(-r)\,dr\right)\\ &= \int_0^t g'(t-r)p_s^\beta(r)\,dr \pm g(0)p_s^\beta(t).\end{aligned}$$

Then $C^1[0,T]$ is a dense subspace of $\mathrm{Dom}(\mathcal{L}_\beta)$ which is invariant under P^β, and so it is a core for $(\mathcal{L}_\beta, \mathrm{Dom}(\mathcal{L}_\beta))$ by [Böttcher et al., 2013, Lemma 1.34].

(ii) Similarly to part (i), it can be shown that $P_s^{\beta,\mathrm{kill}} f(t) = \int_0^t f(r)p_s^\beta(t-r)\,dr$. To show $\mathrm{Dom}(\mathcal{L}_\beta)\cap\{f(0)=0\} \subset \mathrm{Dom}(\mathcal{L}_\beta^{\mathrm{kill}})$, let $f \in \mathrm{Dom}(\mathcal{L}_\beta)\cap\{f(0)=0\}$, then for some $\lambda>0$, let $g \in C[0,T]$ such that

$$f(t) = \int_0^\infty e^{-\lambda s}P_s^\beta g(t)\,ds, \quad \text{then} \quad g(0)\frac{1}{\lambda} = \int_0^\infty e^{-\lambda s}P_s^\beta g(0)\,ds = f(0) = 0,$$

and so $g \in C_0[0,T]$. As $P_s^\beta = P_s^{\beta,\mathrm{kill}}$ on $C_0[0,T]$, it follows that $f \in \mathrm{Dom}(\mathcal{L}_\beta^{\mathrm{kill}})$. The inclusion $\mathrm{Dom}(\mathcal{L}_\beta)\cap\{f(0)=0\} \supset \mathrm{Dom}(\mathcal{L}_\beta^{\mathrm{kill}})$ is immediate using $P_s^\beta = P_s^{\beta,\mathrm{kill}}$ on $C_0[0,T]$. By equating a resolvent equation, it follows that $\mathcal{L}_\beta^{\mathrm{kill}} = \mathcal{L}_\beta$ on $\mathrm{Dom}(\mathcal{L}_\beta^{\mathrm{kill}})$. Invariance of $C_0^1[0,T]$ can be proven as in part (i). The last statement now follows from part (i).

(iii) The first identity follows from the third identity (1.10). The second identity follows by [Zolotarev, 1986, Theorem 2.10.2]. To prove the third identity (1.10), recall that

$$p_s^\beta(t-r) = s^{-1/\beta}p_1^\beta(s^{-1/\beta}(t-r)), \quad t>r,$$

and then compute

$$\int_0^\infty p_s^\beta(t,r)\,ds = (t-r)^{\beta-1}\int_0^\infty u^{-1/\beta}p_1^\beta(u^{-1/\beta})\,du = (t-r)^{\beta-1}\frac{1}{\Gamma(\beta)},$$

using the Mellin transform of the β-stable density p_1^β for the last equality (see for example [Zolotarev, 1986, Theorem 2.6.3]).

□

Definition 1.4.13. We say that a bounded open set $\Omega \subset \mathbb{R}^d$ is a *regular set* if Ω satisfies the

[12] We select $c_+ = \Gamma(-\alpha)^{-1}$ and $c_- = 0$ in [Bernyk et al., 2011, Proposition 14]. In the statement of [Bernyk et al., 2011, Proposition 14] it is required that $F \in C^2([0,\infty))$, but $F \in C^1([0,\infty))$ is enough.

exterior cone condition at every point $\partial\Omega$, i.e. for each $x \in \partial\Omega$ there exists a finite right circular open cone V_x with vertex x, such that $V_x \subset \Omega^c$ (see [Chen et al., 2012, end of Section 4]).

Remark 1.4.14. In Chapter 4 the set Ω is will always be a regular set.

Proposition 1.4.15. Let Ω be a regular set. Define the sub-process $X_\Omega^{x,\alpha}$ started at $x \in \Omega$ by

$$X_\Omega^{x,\alpha}(s) := \begin{cases} X^{x,\alpha}(s), & s < \tau_\Omega^\alpha(x), \\ \text{cemetery}, & s \geq \tau_\Omega^\alpha(x), \end{cases}$$

(i) Then $X_\Omega^{x,\alpha}$ induces a sub-Feller semigroup on $C_{\partial\Omega}(\Omega)$, which we denote by $P^\alpha = \{P_s^\alpha\}_{s\geq 0}$, and we denote its generator by $(\mathcal{L}_\alpha, \mathrm{Dom}(\mathcal{L}_\alpha))$. Moreover if $u \in \mathrm{Dom}(\mathcal{L}_\alpha)$ then there exists a sequence $u_n \in C_{\partial\Omega}(\Omega) \cap C^2(\Omega)$ such that $u_n \to u$ uniformly and $\Delta_\Omega^{\frac{\alpha}{2}} u_n \to \mathcal{L}_\alpha u$ uniformly on compact subsets of Ω. The transition density of $X_\Omega^{x,\alpha}(s)$, denoted by $P_s^\alpha(x,y)$, is jointly continuous in x and y, for every $s > 0$.

(ii) For every $u \in \mathrm{Dom}(\mathcal{L}_\alpha)$ and $\varphi \in C_c^2(\Omega)$ it holds

$$\int_\Omega \mathcal{L}_\alpha u \varphi \, dx = \int_\Omega u \Delta_\Omega^{\frac{\alpha}{2}} \varphi \, dx. \tag{1.11}$$

(iii) The semigroup P^α induces a strongly continuous contraction semigroup on $L^2(\Omega)$, and we denote its generator by $(\mathcal{L}_{\alpha,2}, \mathrm{Dom}(\mathcal{L}_{\alpha,2}))$. Moreover there exists a sequence of positive numbers $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$, and an orthonormal basis $\{\psi_n\}_{n\in\mathbb{N}}$ of $L^2(\Omega)$, so that $P_s^\alpha \psi_n = e^{-\lambda_n s}\psi_n$ in $L^2(\Omega)$, for every $n \in \mathbb{N}$, $s > 0$. For $k \geq 1$, we denote by $\mathrm{Dom}(\mathcal{L}_{\alpha,2}^k)$ the subset of $L^2(\Omega)$ such that $\|f\|_{\mathcal{L}_{\alpha,2}^k} := \left(\sum_{n=1}^\infty \lambda_n^{2k} \langle f, \psi_n\rangle^2\right)^{1/2} < \infty$. Moreover, P^α on $C_{\partial\Omega}(\Omega)$ has the same set of eigenvalues and eigenfunctions as P^α on $L^2(\Omega)$.

Proof. (i) The first two statements are a consequence of [Baeumer et al., 2016b, Lemma 2.2 and Theorem 2.7][13]. The last statement follows by the strong Markov property along with joint continuity of the transition densities of $X^{x,\alpha}$ (see for example [Chen et al., 2012, Section 4]).

(ii) The operator $\Delta_\Omega^{\frac{\alpha}{2}}$ is self-adjoint in the sense that

$$\int_\Omega \Delta_\Omega^{\frac{\alpha}{2}} u \varphi \, dx = \int_\Omega u \Delta_\Omega^{\frac{\alpha}{2}} \varphi \, dx, \tag{1.12}$$

if $\varphi \in C_c^2(\Omega)$ and $u \in C_{\partial\Omega}(\Omega) \cap C^2(\Omega)$. Now use the approximating sequence from part (i) of the current proposition to conclude.

(iii) These results can be found in [Chen et al., 2012, Section 4] and references therein. □

1.5 Time-nonlocal and time-fractional EEs

This section is a guide for the reader, where we give quick overview of which EEs we treat, and how we do so.

[13] Connectedness of Ω is not actually used for these two statements.

- In Chapter 4 we give sufficient conditions for existence and uniqueness of classical solutions for the Marchaud time-fractional EE

$$\begin{cases} D^{\beta}_{\infty}u(t,x) = \Delta_{\Omega}^{\frac{\alpha}{2}}u(t,x) + g(t,x), & \text{in } (0,T]\times\Omega, \\ u(t,x) = 0, & \text{in } (0,T]\times\partial\Omega, \\ u(t,x) = \phi(t,x), & \text{in } (-\infty,0]\times\Omega, \end{cases} \tag{1.13}$$

  for $\Omega \subset \mathbb{R}^d$ a regular domain, and we prove that u allows the stochastic representation (1.2) (Theorem 4.3.6). This EE includes the inhomogeneous Caputo EE

$$\begin{cases} D^{\beta}_{0}u(t,x) = \Delta_{\Omega}^{\frac{\alpha}{2}}u(t,x) + g(t,x), & \text{in } (0,T]\times\Omega, \\ u(t,x) = 0, & \text{in } (0,T]\times\partial\Omega, \\ u(t,x) = \phi_0(x), & \text{in } \{0\}\times\Omega; \end{cases} \tag{1.14}$$

  which can be seen by selecting $\phi(t) = \phi(0)$ for all $t < 0$ in (1.13) and (1.2).

- In Chapter 2 we prove existence and uniqueness for generalised solutions for the inhomogeneous Caputo-type EE

$$\begin{cases} D^{(\nu)}_{0}u(t,x) = \mathcal{L}_{\Omega}u(t,x) + g(t,x), & \text{in } (0,T]\times\Omega, \\ u(t,x) = 0, & \text{in } (0,T]\times\partial\Omega, \\ u(0,x) = \phi_0(x), & \text{in } \{0\}\times\Omega, \end{cases} \tag{1.15}$$

  along with the representation (1.2) under assumptions (H0) and (H1a) on the kernel ν, and assumption (H1b) on $(\mathcal{L}_{\Omega}, \mathrm{Dom}(\mathcal{L}_{\Omega}))$ (Theorem 2.1.4).

- In Chapter 3, we exploit the generalised solutions of Chapter 2 to prove existence of weak solutions of the Marchaud-type evolution equation

$$\begin{cases} D^{(\nu)}_{\infty}u(t,x) = \Delta u(t,x) + f(t,x), & \text{in } (0,T]\times\Omega, \\ u(t,x) = 0, & \text{in } (0,T]\times\partial\Omega, \\ u(t,x) = \phi(t,x), & \text{in } (-\infty,0]\times\Omega, \end{cases} \tag{1.16}$$

  along with the representation (1.2), where Δ is the Dirichlet Laplacian for a regular domain Ω and we assume (H0') and (H1a') on the kernel ν (Theorem 3.2.10).

1.6 Space-time Feller semigroups

This section is dedicated to the proof of Theorem 1.6.3, which we use to treat evolution equations as elliptic boundary value problems. As this theorem is rather technical, in Section 1.6.1 we provide a formal example with the heat equation in order to explain how we use it.

Remark 1.6.1. The abstract operators $\mathcal{L}_{(\nu)}$ and $\mathcal{L}^{\infty}_{(\nu)}$ will be the semigroup counterparts of our nonlocal time-derivatives $-D^{(\nu)}_{0}$ and $-D^{(\nu)}_{\infty}$, respectively.

Remark 1.6.2. We now show that the closure of, say $\mathcal{L}_{(\nu)} + \mathcal{L}_\Omega$ on a Stone-Weierstrass type of set, is the generator $\mathcal{L}$ of the product semigroup $s \mapsto P_s^{(\nu)} P_s^\Omega$. Then, formally, the potential $u = (-\mathcal{L})^{-1} g$ solves the EE $\mathcal{L}_{(\nu)} u = -\mathcal{L}_\Omega u - g$. It turns out that this theorem provides a rather general setting to begin our study of solutions for EEs.

Theorem 1.6.3. *With the notation of Definition 1.3.10 and Definition 1.3.14, let $\mathcal{C}_{(\nu)}^\infty$, $\mathcal{C}_{(\nu)}$, $\mathcal{C}_{(\nu)}^{kill}$ and $\mathcal{C}_\Omega$ be invariant cores for $(\mathcal{L}_{(\nu)}^\infty, Dom(\mathcal{L}_{(\nu)}^\infty))$, $(\mathcal{L}_{(\nu)}, Dom(\mathcal{L}_{(\nu)}))$, $(\mathcal{L}_{(\nu)}^{kill}, Dom(\mathcal{L}_{(\nu)}^{kill}))$ and $(\mathcal{L}_\Omega, Dom(\mathcal{L}_\Omega))$, respectively.*

(i) Then $P^{(\nu),\Omega} = \{P_s^{(\nu)} P_s^\Omega\}_{s\geq 0}$ is a Feller semigroup on $C_{\partial\Omega}([0,T]\times\Omega)$. The generator $(\mathcal{L}_{(\nu),\Omega}, Dom(\mathcal{L}_{(\nu),\Omega}))$ of $P^{(\nu),\Omega}$ is the closure of

$$\Big(\mathcal{L}_{(\nu)} + \mathcal{L}_\Omega, Span\left\{\mathcal{C}_{(\nu)}\cdot\mathcal{C}_\Omega\right\}\Big) \quad in \quad C_{\partial\Omega}([0,T]\times\Omega),$$

where $P^{(\nu)}$ and $\mathcal{L}_{(\nu)}$ act on the $[0,T]$-variable, and P^Ω and $\mathcal{L}_\Omega$ act on the Ω-variable.

(ii) Then $P^{(\nu),\Omega,kill} = \{P_s^{(\nu),kill} P_s^\Omega\}_{s\geq 0}$ is a Feller semigroup on $C_{0,\partial\Omega}([0,T]\times\Omega)$. The generator $(\mathcal{L}_{(\nu),\Omega}^{kill}, Dom(\mathcal{L}_{(\nu),\Omega}^{kill}))$ of $P^{(\nu),\Omega,kill}$ is the closure of

$$\Big(\mathcal{L}_{(\nu)}^{kill} + \mathcal{L}_\Omega, Span\{\mathcal{C}_{(\nu)}^{kill}\cdot\mathcal{C}_\Omega\}\Big) \quad in \quad C_{0,\partial\Omega}([0,T]\times\Omega),$$

where $P^{(\nu),kill}$ and $\mathcal{L}_{(\nu)}^{kill}$ act on the $[0,T]$-variable, and P^Ω and $\mathcal{L}_\Omega$ act on the Ω-variable.

(iii) Then $P^{(\nu),\Omega,\infty} = \{P_s^{(\nu),\infty} P_s^\Omega\}_{s\geq 0}$ is a Feller semigroup on $C_{\infty,\partial\Omega}((-\infty,T]\times\Omega)$. The generator $(\mathcal{L}_{(\nu),\Omega}^\infty, Dom(\mathcal{L}_{(\nu),\Omega}^\infty))$ of $P^{(\nu),\Omega,\infty}$ is the closure of

$$\Big(\mathcal{L}_{(\nu)}^\infty + \mathcal{L}_\Omega, Span\{\mathcal{C}_{(\nu)}^\infty\cdot\mathcal{C}_\Omega\}\Big) \quad in \quad C_{\infty,\partial\Omega}((-\infty,T]\times\Omega),$$

where $P^{(\nu),\infty}$ and $\mathcal{L}_{(\nu)}^\infty$ act on the $(-\infty,T]$-variable, and P^Ω and $\mathcal{L}_\Omega$ act on the Ω-variable.

(iv) It holds that

$$P_s^{(\nu),\Omega} = P_s^{(\nu),\Omega,kill} \text{ on } C_{0,\partial\Omega}([0,T]\times\Omega), \quad \mathcal{L}_{(\nu),\Omega} = \mathcal{L}_{(\nu),\Omega}^{kill} \text{ on } Dom(\mathcal{L}_{(\nu),\Omega}^{kill}),$$

and

$$Dom(\mathcal{L}_{(\nu),\Omega}^{kill}) = Dom(\mathcal{L}_{(\nu),\Omega}) \cap \{f(0)=0\}.$$

Proof. The proofs of (i), (ii) and (iii) are very similar. We therefore prove only (ii) for Ω bounded. The case $\Omega = \mathbb{R}^d$ is simpler and omitted.

Note that $P_s^{(\nu),\text{kill}} P_r^\Omega = P_r^\Omega P_s^{(\nu),\text{kill}}$ for every $s, r \geq 0$, and that

$$\|P_s^\Omega f\|_{C([0,T]\times\bar\Omega)} \leq \|f\|_{C([0,T]\times\bar\Omega)}, \quad \|P_s^{(\nu),\text{kill}} f\|_{C([0,T]\times\bar\Omega)} \leq \|f\|_{C([0,T]\times\bar\Omega)},$$

for every $f \in C_{0,\partial\Omega}([0,T]\times\Omega)$, $s \geq 0$. It is then easy to prove that $P^{(\nu),\Omega,\text{kill}}$ is Feller semigruop on $C_{0,\partial\Omega}([0,T]\times\Omega)$. We denote the generator of $P^{(\nu),\Omega,\text{kill}}$ by $(\mathcal{L}_{(\nu),\Omega}^{\text{kill}}, \text{Dom}(\mathcal{L}_{(\nu),\Omega}^{\text{kill}}))$. Let $f = pq$, where $p \in \mathcal{C}^{\mathrm{kill}}_{(\nu)}$ and $q \in \mathcal{C}_\Omega$. Then, by a standard triangle inequality argument, we obtain

$$\begin{aligned}
\Big| \frac{P^{(\nu),\mathrm{kill}}_h P^\Omega_h f(t,x) - f(t,x)}{h} &- (\mathcal{L}^{\mathrm{kill}}_{(\nu)} + \mathcal{L}_\Omega) f(t,x) \Big| \\
&\leq \|p\|_{C[0,T]} \left\| \frac{P^\Omega_h q - q}{h} - \mathcal{L}_\Omega q \right\|_{C(\bar\Omega)} + \|\mathcal{L}_\Omega q\|_{C(\bar\Omega)} \left\| P^{(\nu),\mathrm{kill}}_h p - p \right\|_{C[0,T]} \\
&\quad + \|q\|_{C(\bar\Omega)} \left\| \frac{P^{(\nu),\mathrm{kill}}_h p - p}{h} - \mathcal{L}^{\mathrm{kill}}_{(\nu)} p \right\|_{C[0,T]} \to 0,
\end{aligned}$$

as $h \downarrow 0$. An induction argument proves that $\mathrm{Span}\{\mathcal{C}^{\mathrm{kill}}_{(\nu)} \cdot \mathcal{C}_\Omega\} \subset \mathrm{Dom}(\mathcal{L}^{\mathrm{kill}}_{(\nu),\Omega})$ and $\mathcal{L}^{\mathrm{kill}}_{(\nu),\Omega} = (\mathcal{L}^{\mathrm{kill}}_{(\nu)} + \mathcal{L}_\Omega)$ on $\mathrm{Span}\{\mathcal{C}^{\mathrm{kill}}_{(\nu)} \cdot \mathcal{C}_\Omega\}$. Observing that $\mathrm{Span}\{\mathcal{C}^{\mathrm{kill}}_{(\nu)} \cdot \mathcal{C}_\Omega\}$ is invariant under $P^{(\nu),\Omega,\mathrm{kill}}$ and it is a subspace of $\mathrm{Dom}(\mathcal{L}^{\mathrm{kill}}_{(\nu),\Omega})$, if we can prove that $\mathrm{Span}\{\mathcal{C}^{\mathrm{kill}}_{(\nu)} \cdot \mathcal{C}_\Omega\}$ is dense in $C_{0,\partial\Omega}([0,T]\times\Omega)$, we are done by [Böttcher et al., 2013, Lemma 1.34]. So proceed by noting that set $\mathrm{Span}\{C^\infty([0,T]) \cdot C^\infty(\bar\Omega)\}$ is a sub-algebra of $C([0,T] \times \bar\Omega)$ that contains constant functions and separates points. Hence $\mathrm{Span}\{C^\infty([0,T]) \cdot C^\infty(\bar\Omega)\}$ is dense in $C([0,T] \times \bar\Omega)$ by Stone-Weierstrass Theorem for compact[14] Hausdorff spaces. We now prove density of the following set

$$\mathrm{Span}\{C^\infty_c(0,T] \cdot C^\infty_c(\Omega)\} \subset C_{0,\partial\Omega}([0,T] \times \Omega).$$

For $f \in C_{0,\partial\Omega}([0,T] \times \Omega)$ we take a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathrm{Span}\{C^\infty([0,T]) \cdot C^\infty(\bar\Omega)\}$ such that $f_n \to f$, where $f_n(t,x) = \sum_{i=1}^{N_n} p_{i,n}(t) q_{i,n}(x)$, for some $N_n \in \mathbb{N}$ depending on $n \in \mathbb{N}$. Let $1_{T,n} \in C^\infty_c(0,T]$ and $1_{\Omega,n} \in C^\infty_c(\Omega)$ be smooth functions for each $n \in \mathbb{N}$, such that $0 \leq 1_{T,n}, 1_{\Omega,n} \leq 1$, $1_{T,n}(t) = 1_{\Omega,n}(x) = 1$ for $t \in (\frac{1}{n}, T]$ and $x \in K_n$, and $1_{T,n}(t) = 1_{\Omega,n}(x) = 0$ for $t \in (0, \frac{1}{n+1}]$ and $x \in \Omega \backslash K_{n+1}$, where K_n is compact, $K_n \subset K_{n+1} \subset \Omega$ for each n, and $\cup_n K_n = \Omega$. Define for each $n \in \mathbb{N}$,

$$(t,x) \mapsto \tilde f_n(t,x) := \sum_{i=1}^{N_n} p_{i,n}(t) 1_{T,n}(t) q_{i,n}(x) 1_{\Omega,n}(x) \in \mathrm{Span}\{C^\infty_c(0,T] \cdot C^\infty_c(\Omega)\}.$$

Then, as $n \to \infty$

$$\begin{aligned}
\|\tilde f_n - f\|_{C([0,T]\times\Omega)} &\leq \|f_n - f\|_{C([\frac{1}{n},T]\times K_n)} + \|\tilde f_n - f\|_{C\left((\frac{1}{n+1},\frac{1}{n}]\times\bar\Omega \cup [0,T]\times K_{n+1}\backslash K_n\right)} \\
&\quad + \|f\|_{C\left([0,T]\times\bar\Omega\backslash K_{n+1} \cup [0,\frac{1}{n+1}]\times\bar\Omega\right)} \to 0.
\end{aligned}$$

As in general $C^\infty_c(\Omega) \not\subset \mathrm{Dom}(\mathcal{L}_\Omega)$, we need to work a bit more. For any $u \in C_{0,\partial\Omega}([0,T] \times \Omega)$ we can now take a uniformly approximating sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{Span}\{C^\infty_c(0,T] \cdot C^\infty_c(\Omega)\}$. Denote $u_n(t,x) = \sum_{i=1}^{N_n} p_{i,n}(t) q_{i,n}(x)$, for some $N_n \in \mathbb{N}$ depending on $n \in \mathbb{N}$, where $p_{i,n} \in C^\infty_c(0,T]$, $q_{i,n} \in C^\infty_c(\Omega)$ are non-zero, for each $i \in \{1,...,N_n\}$, $n \in \mathbb{N}$. As $\mathcal{C}_{(\nu)}$ and $\mathcal{C}_\Omega$ are dense in $C_0([0,T]) \supset C^\infty_c(0,T]$ and $C_{\partial\Omega}(\Omega) \supset C^\infty_c(\Omega)$, respectively, we can pick $\{(\tilde p_{i,n}, \tilde q_{i,n}) : i \in \{1,...,N_n\}, n \in \mathbb{N}\} \subset \mathcal{C}_{(\nu)} \times \mathcal{C}_\Omega$, in the following fashion: for each triplet $(N_n, p_{i,n}, q_{i,n})$, first pick

[14] In the case of unbounded domains (part (iii) of the current lemma) use the Stone-Weierstrass Theorem for locally compact Hausdorff spaces.

$\tilde{p}_{i,n}$ so that

$$\|p_{i,n} - \tilde{p}_{i,n}\|_{C[0,T]} \leq \frac{1}{nN_n\|q_{i,n}\|_{C(\bar{\Omega})}},$$

secondly pick $\tilde{q}_{i,n}$ so that

$$\|q_{i,n} - \tilde{q}_{i,n}\|_{C(\bar{\Omega})} \leq \frac{1}{nN_n\|\tilde{p}_{i,n}\|_{C[0,T]}}.$$

Then, after defining $\tilde{u}_n(t,x) := \sum_{i=1}^{N_n} \tilde{p}_{i,n}(t)\tilde{q}_{i,n}(x)$, we obtain

$$\begin{aligned}
\|u - \tilde{u}_n\|_\infty &\leq \|u - u_n\|_\infty + \|u_n - \tilde{u}_n\|_\infty \\
&\leq \|u - u_n\|_\infty + \sum_{i=1}^{N_n} \|p_{i,n}q_{i,n} - \tilde{p}_{i,n}\tilde{q}_{i,n}\|_\infty \\
&\leq \|u - u_n\|_\infty + \sum_{i=1}^{N_n} (\|q_{i,n}\|_\infty\|p_{i,n} - \tilde{p}_{i,n}\|_\infty + \|\tilde{p}_{i,n}\|_\infty\|q_{i,n} - \tilde{q}_{i,n}\|_\infty) \\
&\leq \|u - u_n\|_\infty + \sum_{i=1}^{N_n} \left(\frac{\|q_{i,n}\|_\infty}{nN_n\|q_{i,n}\|_\infty} + \frac{\|\tilde{p}_{i,n}\|_\infty}{nN_n\|\tilde{p}_{i,n}\|_\infty}\right) \\
&= \|u - u_n\|_\infty + \sum_{i=1}^{N_n} \frac{2}{nN_n} \\
&\leq \|u - u_n\|_\infty + \frac{2}{n} \to 0, \qquad \text{as } n \to \infty.
\end{aligned}$$

(iv) The first claim is an immediate consequence of $P^{(\nu),\mathrm{kill}} = P^{(\nu)}$ on $C_0[0,T]$. The second claim follows from the third by considering a resolvent equation. To prove the third claim, we show the equivalent statement

$$\mathrm{Dom}(\mathcal{L}^{\mathrm{kill}}_{(\nu),\Omega}) \subset \mathrm{Dom}(\mathcal{L}_{(\nu),\Omega}), \quad \text{and} \quad \text{if } u \in \mathrm{Dom}(\mathcal{L}_{(\nu),\Omega}), \text{ then } u - u(0) \in \mathrm{Dom}(\mathcal{L}^{\mathrm{kill}}_{(\nu),\Omega}).$$

The first inclusion is immediate using $P_s^{(\nu),\Omega} = P_s^{(\nu),\Omega,\mathrm{kill}}$, on $C_{0,\partial\Omega}([0,T]\times\Omega)$. For the second part, let $u \in \mathrm{Dom}(\mathcal{L}_{(\nu),\Omega})$ and consider its resolvent representation for some $\lambda > 0$ and $g \in C_{\partial\Omega}([0,T]\times\Omega)$. Then

$$u(0,x) = \int_0^\infty e^{-\lambda s} P_s^{(\nu)} P_s^{\Omega} g(0,x)\, ds = \int_0^\infty e^{-\lambda s} P_s^{(\nu)} P_s^{\Omega}(g(0))(t,x)\, ds,$$

as $P_s^{(\nu)} g(0,x) = P_s^{(\nu)}(g(0))(t,x)$. Now consider

$$\begin{aligned}
u(t,x) - u(0,x) &= \int_0^\infty e^{-\lambda s} P_s^{\Omega} P_s^{(\nu)}(g - g(0))(t,x)\, ds \\
&= \int_0^\infty e^{-\lambda s} P_s^{\Omega} P_s^{(\nu),\mathrm{kill}}(g - g(0))(t,x)\, ds \in \mathrm{Dom}(\mathcal{L}^{\mathrm{kill}}_{(\nu),\Omega}),
\end{aligned}$$

where we use the fact that $P^{(\nu),\mathrm{kill}} = P^{(\nu)}$ on $C_{0,\partial\Omega}([0,T]\times\Omega)$ and that $g - g(0) \in C_{0,\partial\Omega}([0,T]\times\Omega)$. $\square$

Remark 1.6.4. Note that

$$(-\mathcal{L}^{\text{kill}}_{(\nu),\Omega})^{-1}g(t,x) = \int_0^\infty P^{(\nu),\Omega}_s g(t,x)ds = \mathbf{E}\left[\int_0^{\tau_0(t)\wedge\tau_\Omega(x)} g(-X^{t,(\nu)}(s), X^{x,\Omega}(s))\, ds\right],$$

for $g \in C_{0,\partial\Omega}([0,T]\times\Omega)$. Also, from now on we might write $\tau_{t,x}$ for $\tau_0(t)\wedge\tau_\Omega(x)$.

Remark 1.6.5. Theorem 1.6.3-(iv), although unsurprising, is a vital technical ingredient for this work. This is because it allows to obtain uniqueness of our notion of a *solution in the domain of the generator* for EEs (see the proof of Theorem 2.1.4-(i)). Such notion of solution is our building block for weak solutions to EEs.

The next Proposition will be only used in Section 2.2.2.

Proposition 1.6.6. Suppose that $\mathcal{L}_\Omega$ is bounded. Then, under the assumptions of Theorem 1.6.3-(ii), $f \in \text{Dom}(\mathcal{L}^{\text{kill}}_{(\nu),\Omega})$ implies that $f(\cdot,x) \in \text{Dom}(\mathcal{L}_{(\nu)})$ for each $x \in \Omega$. In particular $\mathcal{L}_{(\nu),\Omega}f = (\mathcal{L}_{(\nu)} + \mathcal{L}_\Omega)f$ on $(0,T]\times\Omega$.

Proof. Let $f \in \text{Dom}(\mathcal{L}^{\text{kill}}_{(\nu),\Omega})$. Then by Theorem 1.6.3-(ii) there exists $\{f_n\}_{n\in\mathbb{N}} \subset \text{Span}\{\text{Dom}(\mathcal{L}^{\text{kill}}_{(\nu)})\cdot \text{Dom}(\mathcal{L}_\Omega)\}$ such that $f_n \to f$ and $(\mathcal{L}^{\text{kill}}_{(\nu)} + \mathcal{L}_\Omega)f_n = \mathcal{L}^{\text{kill}}_{(\nu),\Omega}f_n \to \mathcal{L}^{\text{kill}}_{(\nu),\Omega}f$, uniformly . As $\mathcal{L}_\Omega$ is bounded $\mathcal{L}_\Omega f_n \to \mathcal{L}_\Omega f$ and so $\{\mathcal{L}_\Omega f_n\}_{n\in\mathbb{N}}$ is Cauchy in $C_{0,\partial\Omega}([0,T]\times\Omega)$. Clearly for each $x \in \Omega$ $f_n(\cdot,x) \to f(\cdot,x)$ in $C_0[0,T]$, and by the definition of $\text{Span}\{\text{Dom}(\mathcal{L}^{\text{kill}}_{(\nu)})\cdot\text{Dom}(\mathcal{L}_\Omega)\}$ $f_n(\cdot,x) \in \text{Dom}(\mathcal{L}^{\text{kill}}_{(\nu)})$ for each $n\in\mathbb{N}$. If we show that for each $x\in\Omega$ the sequence $\mathcal{L}^{\text{kill}}_{(\nu)}f_n(\cdot,x)$ is Cauchy in $C_0[0,T]$ we are done as $\mathcal{L}^{\text{kill}}_{(\nu)}$ is a closed operator on $C_0[0,T]$. This follows from the inequality

$$\left|(\mathcal{L}^{\text{kill}}_{(\nu)}f_n - \mathcal{L}^{\text{kill}}_{(\nu)}f_m)(t,x)\right| \le \|\mathcal{L}^{\text{kill}}_{(\nu),\Omega}f_n - \mathcal{L}^{\text{kill}}_{(\nu),\Omega}f_m\|_\infty + \|\mathcal{L}_\Omega f_n - \mathcal{L}_\Omega f_m\|_\infty,$$

and by taking n and m large. □

1.6.1 Example: heat equation

We give a formal example to illustrate our boundary value problem viewpoint in the case of the standard heat equation

$$\begin{cases} \left(-\dfrac{\partial_t}{\kappa} + \dfrac{\Delta}{2}\right)u = 0, & \text{on } \Gamma := (0,T]\times\mathbb{R}^d, \quad \kappa > 0, \\ u = \phi(0), & \text{on } \partial\Gamma := \{0\}\times\mathbb{R}^d, \end{cases} \tag{1.17}$$

and then we mimic our future use of Theorem 1.6.3 in this basic context.

Observe that $(-\partial_t/\kappa, C^1[0,T])$ is the generator of the Feller semigroup corresponding to the decreasing drift absorbed at 0

$$-X^{t,\kappa}(s) := \begin{cases} t - s/\kappa, & \text{if } t > s/\kappa, \\ 0, & \text{if } t \le s/\kappa. \end{cases}$$

Then the first exit time is given by

$$\tau_0(t) := \inf\{s > 0 : -X^{t,\kappa}(s) \leq 0\} = \kappa t, \quad \text{a.s., for each } t \geq 0.$$

On the other hand $(\Delta/2, C^2_\infty(\mathbb{R}^d))$ generates the standard Brownian motion on $\mathbb{R}^d$. Now, Theorem 1.6.3, suggests that $(-\partial_t/\kappa + \Delta/2)$ generates the $[0,T] \times \mathbb{R}^d$-valued Feller process $(-X^{t,\kappa}(s), x + B(s))$, whose first exit time from the 'domain' $(0,T] \times \mathbb{R}^d$ is $\tau_0(t)$. Then, the natural guess for the Feynman-Kac formula for the boundary value problem (1.17) is

$$\mathbf{E}[\phi(t - \tau_0(t)/\kappa, x + B(\tau_0(t)))] = \mathbf{E}[\phi(0, x + B(\kappa t))],$$

the well known solution to the heat equation (1.17).

The main technical issue to perform the above steps rigorously is that the potential $(\partial_t/\kappa - \Delta/2)^{-1}$ for $f \in C_\infty([0,T] \times \mathbb{R}^d)$ is in general not defined, as

$$(\partial_t/\kappa - \Delta/2)^{-1} f(t,x) = \int_0^{\kappa t} \mathbf{E}[f(s, x + B(s))]\, ds + \int_{\kappa t}^\infty \mathbf{E}[f(0, x + B(s))\, ds,$$

and generally $\int_{\kappa t}^\infty \mathbf{E}[f(0, x + B(s))]\, ds = \infty$. But instead $(\partial_t/\kappa - \Delta/2)^{-1} : C_{0,\infty}([0,T] \times \mathbb{R}^d) \to C_{0,\infty}([0,T] \times \mathbb{R}^d)$ is bounded[15], which implies that the inverse exists and it is a bijection from $C_{0,\infty}([0,T] \times \mathbb{R}^d)$ to the domain of the generator $(-\partial_t/\kappa + \Delta/2)$, allowing for a wellposed notion of solution. In order to recover non-zero initial conditions, we solve (1.17) for the forcing term $-f - \Delta\phi(0)$, for any $f \in C_\infty([0,T] \times \mathbb{R}^d)$ such that $f(0) = \Delta\phi(0)$. Then the inverse is

$$(\partial_t/\kappa - \Delta/2)^{-1}(f + \Delta\phi(0)) = \int_0^{\kappa t} \mathbf{E}[f(s, x + B(s))]\, ds + \int_0^{\kappa t} \mathbf{E}[\Delta\phi(0, x + B(s))]\, ds.$$

Then we *shift the inverse up*[16] by $\phi(0)$, so that

$$u(t,x) := \phi(0) + (\partial_t/\kappa - \Delta/2)^{-1}(f + \Delta\phi(0)) = \int_0^{\kappa t} \mathbf{E}[f(s, x + B(s))]\, ds + \mathbf{E}[\phi(0, x + B(\kappa t))],$$

by Dynkin formula. Note that $(\partial_t/\kappa - \Delta/2)u = f$ and $u(0) = \phi(0)$.
To conclude we take a uniformly bounded sequence of forcing terms f_n going to 0 a.e. such that $f_n(0) = \Delta\phi(0)$ for each n, and we pass to the limit[17]. Such (unique) pointwise limit will be our notion generalised solution.

Briefly, to construct weak solutions, we start with solutions in the domain of the generator. In such case we show that they satisfy an appropriate dual pairing of the kind

$$\langle \mathcal{L}_{(\nu),\Omega} u, \varphi \rangle = \langle u, (-D_0^{(\nu),*} + \mathcal{L}^*_\Omega)\varphi \rangle,$$

[15] Here is where we need the killed version of the space-time process constructed in Theorem 1.6.3-(ii).
[16] Here we use Theorem 1.6.3-(iv), as it allows to preserve the uniqueness of the notion of solution in the domain of the generator after the shift.
[17] To perform this step we will use assumptions (H1a) and (H1b).

using the Stone-Weierstrass core $\mathrm{Span}\left\{C^1[0,T]\cdot \mathrm{Dom}(\mathcal{L}_\Omega)\right\}$ from Theorem 1.6.3-(i). We conclude exploiting simple convergence properties of solutions in the domain of the generator to generalised solutions.

Chapter 2

Caputo-type EE; stochastic generalised solution

The main purpose of this chapter[1] is to prove wellposedness and stochastic representation for the solutions of the linear evolution equation (EE)

$$\begin{cases} D_0^{(\nu)} u(t,x) = \mathcal{L}_\Omega u(t,x) + g(t,x), & \text{in } (0,T] \times \Omega, \\ \quad u(t,x) = 0, & \text{in } (0,T] \times \partial\Omega, \\ \quad u(0,x) = \phi_0(x), & \text{in } \Omega, \end{cases} \tag{2.1}$$

and the nonlinear EE

$$\begin{cases} D_0^{(\nu)} u(t,x) = \mathcal{L}_\Omega u(t,x) + f(t,x,u(t,x)), & \text{in } (0,T] \times \Omega, \\ \quad u(t,x) = 0, & \text{in } (0,T] \times \partial\Omega, \\ \quad u(0,x) = \phi_0(x), & \text{in } \Omega, \end{cases} \tag{2.2}$$

where

$$D_0^{(\nu)} u(t) = \int_0^t (u(t) - u(t-r))\nu(t,r)\, dr + (u(t) - u(0)) \int_t^\infty \nu(t,r)\, dr$$

is a generalised differential operator of Caputo-type of order less than 1 acting on the time variable $t \in (0,T]$ (as introduced in Kolokoltsov [2015]), $\mathcal{L}_\Omega$ is the generator of a Feller semigroup on $C_{\partial\Omega}(\Omega)$ acting on space, for $\Omega = \mathbb{R}^d$ or a bounded domain, $\phi_0 \in \mathrm{Dom}(\mathcal{L}_\Omega)$ the domain of the $\mathcal{L}_\Omega$, $g : (0,T] \times \Omega \to \mathbb{R}$ is a bounded measurable function, and $f : (0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a non-linear function satisfying a certain Lipschitz condition.
Note that the Lévy-type kernel ν is allowed to depend on time.

Since Caputo derivatives of order $\beta \in (0,1)$ are special cases of the operators $D_0^{(\nu)}$, the evolution equations in (2.1)-(2.2) include as particular cases a variety of equations studied in the theory of fractional partial differential equations (FPDE's). The latter equations have been successfully used for describing diffusions in disordered media, also called *anomalous diffusions*, which include both *subdiffusions* and *superdiffusions*. Subdiffusion phenomena are usually re-

[1] This chapter contains results published in the joint work Hernández-Hernández et al. [2017], but proved for more general spatial operators $\mathcal{L}_\Omega$.

lated to time-FPDE's, whereas superdiffusions are related to space-FPDE's. We refer, e.g., to Bouchaud and Georges [1990], Carpinteri and Mainardi [1997], Klafter and Sokolov [2005], Mainardi [2001], Mainardi [2010], Kilbas et al. [2006], Meerschaert et al. [2009], Nane [2012], Anh and Leonenko [2001], Kolokoltsov [2008], Meerschaert and Sikorskii [2012], Podlubny [1999], Zaslavsky [2002] Kochubei and Kondratiev [2017] and references cited therein, for an account of historical notes, theory and applications of fractional calculus, as well as different analytical and numerical methods to address both fractional ordinary differential equations (FODE's) and fractional partial differential equations.

In the classical fractional setting, special cases of equation (2.1) include *fractional Cauchy problems*, that is initial value problems of the form

$$\begin{cases} D_0^\beta u(t,x) = \mathcal{L}_\Omega u(t,x), & (t,x) \in [0,T] \times \mathbb{R}^d, \\ u(0,x) = \phi_0(x), & x \in \mathbb{R}^d, \quad \beta \in (0,1), \end{cases} \tag{2.3}$$

where D_0^β stands for the Caputo derivative of order β (acting on the variable t). Equations of the type in (2.3) have been actively studied in the literature. Amongst the standard analytical approaches to solve FPDE's, *the Laplace-Fourier transform* method plays an important role (see, e.g., Diethelm [2010], Edwards et al. [2002], Kilbas et al. [2006], Podlubny [1999], Samko and Marichev [1993], and references therein). From a probabilistic point of view, interesting connections have been found between the solution of time-FPDE's and the transition densities of time-changed Markov processes (see for example Baeumer et al. [2016a], Baeumer et al. [2005], Gorenflo and Mainardi [1998], Kolokoltsov [2008], Kolokoltsov [2011], Meerschaert and Sikorskii [2012], Nonnenmacher [1990]). For instance, a very standard example of the equation (2.3), first studied by Schneider and Wyss Schneider and Wyss [1989] and Kochubei Kochubei [1990] (see also Bouchaud and Georges [1990], Mainardi [2001], Meerschaert and Sikorskii [2012] and references therein), is given by the time-*fractional diffusion equation*, where $\mathcal{L}_\Omega = \frac{1}{2}\Delta$, Δ being the Laplace operator. The work in Baeumer and Meerschaert [2001] provides strong solutions for $\mathcal{L}_\Omega$ being the generator of a strongly continuous contraction semigroup. The work in Leonenko et al. [2013] provides strong solutions for $\mathcal{L}_\Omega$ being the generator of a Pearson diffusion on an interval. In these cases the fundamental solution (or heat kernel) corresponds to the probability density of a self-similar non-Markovian stochastic process, given by the time-changed transition probability function of the diffusion associated with $\mathcal{L}_\Omega$ by the inverse of the β-stable subordinator.

An example of equation (2.3) (with a potential), was studied in Eidelman and Kochubei [2004], wherein the authors determined the fundamental solution of the non-homogeneous Cauchy problem associated with the second-order differential operator with variable coefficients given by

$$\mathcal{L}_\Omega = \sum_{i,j}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{d} b_j(x) \frac{\partial}{\partial x_j} + c(x).$$

The wellposedness of the (abstract) Cauchy problem (2.3) for $\mathcal{L}_\Omega$ being a closed operator in a Banach space was studied in Bazhlekova [1998]. Moreover, evolution equations of the type (2.3) arise, for example, as the limiting evolution of an uncoupled and properly scaled *continuous time* *random walk* (CTRW) with the waiting times in the *domain of attraction of β-stable laws.* This probabilistic model and some of its extensions have been widely studied (see, e.g., Meerschaert and Sikorskii [2012], Scalas [2012], Kolokoltsov [2011], and references therein). The authors in Kolokoltsov and Veretennikova [2014] addressed the regularity of the non-homogeneous time-space fractional linear equation

$$\begin{cases} D_0^\beta u(t,x) = -(-\Delta)^{\alpha/2}u(t,x) + g(t,x), & x \in \mathbb{R}^d,\ t > 0, \\ \quad u(0,x) = \phi_0(x), & x \in \mathbb{R}^d, \end{cases}$$

as well as the wellposedness for the fractional Hamilton-Jacobi-Bellman (HJB) type equation

$$\begin{cases} D_0^\beta u(t,x) = -(-\Delta)^{\alpha/2}u(t,x) + H(t,x,\nabla u(t,x)), & x \in \mathbb{R}^d,\ t > 0, \\ \quad u(0,x) = \phi_0(x), & x \in \mathbb{R}^d, \end{cases}$$

for $\beta \in (0,1)$, $\alpha \in (1,2]$.
Quite related to this chapter is the article Chen [2017][2], which proves stochastic strong solutions for the homogeneous version of (2.1), with time derivative being with $\kappa\partial_t + D_0^{(\nu)}$ for a time independent kernel ν such that $\int_0^\infty \nu(r)\,dr = \infty$. Their technique radically different from ours, being rather direct and it also allows Ω to be a locally compact measure metric space. See also the follow up article Chen et al. [2018]. We also mention the recent arXiv preprints Savov and Toaldo [2018], where ν is allowed spatial dependence, and Kolokoltsov [2017] where $\mathcal{L}_\Omega$ is allowed time dependence.

Using the results presented here, we are able to deduce some of the results known for the previous cases, as well as to extend the analysis to more general situations (see Section 1.3.2 for some possible choices of concrete operators $D_0^{(\nu)}$).
We will first show in Theorem 2.1.4 the wellposedness of problem (2.1), and the stochastic representation for the solution

$$u(t,x) = \mathbf{E}\left[\phi_0\left(X^{x,\Omega}(\tau_0(t))\right)\mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}} + \int_0^{\tau_0(t)\wedge\tau_\Omega(x)} g\left(-X_0^{t,(\nu)}(s), X^{x,\Omega}(s)\right)ds\right]. \tag{2.4}$$

Here $-X_0^{t,(\nu)}$ is the *decreasing* $[0,T]$-valued stochastic process generated by $-D_0^{(\nu)}$ started at $t \in [0,T]$, $\tau_0(t)$ is the first time $-X_0^{t,(\nu)}$ hits $\{0\}$, $X^{x,\Omega}(s)\mathbf{1}_{\{s<\tau_\Omega(x)\}}$ is the stochastic process generated by $\mathcal{L}_\Omega$ started at $x \in \Omega$, and $\tau_\Omega(x)$ is the first exit time of $X^{x,\Omega}$ from Ω. Note that the stochastic representation (2.4) features the (time-changed) process $t \mapsto X^{x,\Omega}(\tau_0(t))$. In the fractional case, $-X_0^{t,(\nu)}(s) = (t - X^\beta(s))\mathbf{1}_{\{s<\tau_0^\beta(t)\}}$ and $\tau_0^\beta(t) = \inf\{s > 0 : X^\beta(s) > t\}$, where X^β is the standard β-stable subordinator, recovering the solution in Baeumer and Meerschaert [2001].
For $\mathcal{L}_\Omega$ bounded and a stronger assumption on the function ν (see assumption (H2)), we will

[2] This article appeared in the same journal release as the content of the current chapter.

give the series representation to the solution of problem (2.1)

$$u(t,x) = \sum_{n=0}^{\infty}((\mathcal{L}_\Omega I_0^{(\nu)})^n \phi_0)(t,x) + \sum_{n=0}^{\infty}((\mathcal{L}_\Omega I_0^{(\nu)})^n I_0^{(\nu)} g)(t,x), \tag{2.5}$$

where $I_0^{(\nu)}$ is the potential operator of the semigroup generated by the (generalised) RL fractional operator $-D_0^{(\nu)}$ (see Theorem 2.2.10). The series in (2.5) provides a generalisation of a certain class of Mittag-Leffler functions. To see this take $\mathcal{L}_\Omega = \lambda \in \mathbb{R}$, and $D_0^{(\nu)} = D_0^\beta$, the Caputo derivative of order $\beta \in (0,1)$, then $I_0^{(\nu)} = I_0^\beta$, the RL fractional integral of order β, and

$$u(t,x) = \phi_0(x)E_\beta(\lambda t^\beta) + \int_0^t g(t-y,x)\beta t^{\beta-1}\frac{d}{dy}E_\beta(\lambda y^\beta)dy,$$

where $E_\beta(z) := \left(\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\beta n+1)}\right)$ (see [Diethelm, 2010, Theorem 7.2] for example). By approximating the generator of a Feller process $\mathcal{L}_\Omega$ with bounded operators (namely the Yosida approximation) we show the convergence of the series representation (2.5) to the stochastic representation (2.4) for the operator $\mathcal{L}_\Omega$ (see Theorem 2.2.16).

As for the non-linear problem (2.2), we study the wellposedness following a similar strategy to the one used for the non-linear equation studied by the authors in Hernández-Hernández and Kolokoltsov [2016]. Namely, by means of the the integral representation (*mild form*) of the solution to the linear problem (2.1), we reduce the analysis of (2.2) to a fixed point problem for a suitable linear operator (see Theorem 2.3.3). Let us mention that, even though in this work we do not include the HJB type case, our results for the generalised non-linear equation (2.2) can be used to extend the wellposedness for the corresponding equations of HJB type.

The results concerning the series representations (2.5) of the solutions to the linear evolution equation (2.1) and the wellposedness of the non-linear evolution equation (2.2) rely on the bounds in Theorem 2.2.4. Theorem 2.2.4 is a consequence of assumption (H2), which implies that for every $t, y \in (0,T]$, $s \in (0,\infty)$, $\mathbf{P}[-X^{t,(\nu)}(s) \geq y] \leq \mathbf{P}[t - X^\beta(s) \geq y]$ where X^β is the β-stable subordinator for some $\beta \in (0,1)$.

Let us briefly describe the two notions of solution used in this work for problem (2.1). We call $u \in C_\infty([0,T]\times\Omega)$ a *solution in the domain of the generator* for problem (2.1) with $g \in C_{\partial\Omega}([0,T]\times\Omega)$, $\phi_0 \in \mathrm{Dom}(\mathcal{L}_\Omega)$, if u satisfies the two equalities in (2.1) and $u \in \mathrm{Dom}(\mathcal{G})$, the domain of the generator $\mathcal{G} = -D_0^{(\nu)} + \mathcal{L}_\Omega$.
This notion of solution is quite natural from the point of view of semigroup theory. To see this consider a strongly continuous semigroup $\{T_s\}_{s\geq 0}$ acting on a Banach space B, let $\mathcal{G}$ be its generator and $\mathrm{Dom}(\mathcal{G})$ the domain of $\mathcal{G}$. Suppose now that the potential operator $(-\mathcal{G})^{-1}$ is bounded on B, then $(-\mathcal{G})^{-1} : B \to \mathrm{Dom}(\mathcal{G})$ is a bijection and $\mathcal{G}(-\mathcal{G})^{-1}g = -g$ (see [Dynkin, 1965, Theorem 1.1']). By viewing problem (2.1) as a Dirichlet problem of the form

$$\mathcal{G}u = -g, \text{ in } (0,T]\times\Omega, \quad u(0) = \phi_0 \text{ in } \{0\}\times\Omega,$$

for $\mathcal{G} = (-D_0^{(\nu)} + \mathcal{L}_\Omega)$, where $-D_0^{(\nu)}$ is the generalised Riemann-Liouville (RL) fractional derivative, $\phi_0 = 0$, we will see that $(-\mathcal{G})^{-1}$ is bounded. From the RL case we extend the definition to the Caputo case. Of course such definition of solution does not allow to choose the boundary condition ϕ_0, as $u(0)$ is determined by the choice of $g \in B$.
The second notion of solution overcomes this issue. Roughly speaking, a function $u \in B([0,T] \times \overline{\Omega})$ is said to be a *generalised solution* to problem (2.1) if u is the point-wise limit of a certain sequence of solutions in the domain of the generator. The stochastic representation of solutions in the domain of the generator allows us to pass to the limit and obtain wellposedness along with the stochastic representation (2.4) of the generalised solution.

This chapter is organized as follows. Section 2.1 focuses on the wellposedness results for the equation (2.1) along with providing the stochastic representation (2.4) for the solution. Section 2.2 introduces the generalised RL integral operator $I_0^{(\nu)}$ and proves that the solution to (2.1) is the limit of a Mittag-Leffler type of series Section 2.3 deals with the wellposedness of the non-linear equation (2.2).

Notation

We use the definitions and notation in Definition 1.3.4, Definition 1.3.10, Definition 1.3.14 and Theorem 1.6.3. In this work we could work with $(\mathcal{L}_\Omega, \mathrm{Dom}(\mathcal{L}_\Omega))$ begin the generatore of a strongly continuous uniformly bounded semigroup on $C_{\partial\Omega}(\Omega)$ (or $C(\overline{\Omega})$ for $\Omega \subset \mathbb{R}^d$ open and bounded), assuming the respective version of assumption (H1b). If $\Omega = \mathbb{R}^d$ the Dirichlet boundary condition in (2.1) becomes void and should be ignored.

2.1 Generalised solution for the inhomogeneous Caputo-type EE

We define our first notion of solution, which is motivated by [Dynkin, 1965, Theorem 1.1'] and Section 1.6.1.

Definition 2.1.1. Let $g \in C_{\partial\Omega}([0,T] \times \Omega)$ and $\phi_0 \in \mathrm{Dom}(\mathcal{L}_\Omega)$ such that $g(0) = -\mathcal{L}_\Omega \phi_0$. We say that a function $u \in C_{\partial\Omega}([0,T] \times \Omega)$ is a *solution in the domain of the generator to problem* (2.1) if

$$\mathcal{L}_{(\nu),\Omega} u = -g \text{ on } (0,T] \times \overline{\Omega}, \quad u(0) = \phi_0, \quad \text{and } u \in \mathrm{Dom}(L_{(\nu),\Omega}). \tag{2.6}$$

The next solution concept for problem (2.1) is defined as a pointwise approximation of solutions in the domain of the generator, such that the approximating data satisfy a dominated convergence type of condition.

Definition 2.1.2. Let $g \in B([0,T] \times \Omega)$ and $\phi_0 \in \mathrm{Dom}(\mathcal{L}_\Omega)$. We say that a function $u \in B([0,T] \times \Omega)$ is a *generalised solution to problem* (2.1) if

$$u = \lim_{n\to\infty} u_n \quad \text{pointwise},$$

where $\{u_n\}_{n\in\mathbb{N}}$ is a sequence of of solutions in the domain of the generator for a corresponding sequence of data $\{g_n\}_{n\in\mathbb{N}} \subset C_{\partial\Omega}([0,T]\times\Omega)$ such that $g_n \to g$ a.e. on $(0,T]\times\Omega$, $\sup_n \|g_n\|_\infty < \infty$, and $g_n(0) = -\mathcal{L}_\Omega \phi_0$ for each $n \in \mathbb{N}$.

Remark 2.1.3. The generalised solution will retain the homogeneous Dirichlet boundary condition on $\partial\Omega$ and the initial condition $u(0) = \phi_0$.

Theorem 2.1.4. *Assume (H0). Then*

(i) If $g + \mathcal{L}_\Omega \phi_0 \in C_{0,\partial\Omega}([0,T] \times \Omega)$ for some $g \in C_{\partial\Omega}([0,T] \times \Omega)$ and $\phi_0 \in Dom(\mathcal{L}_\Omega)$, then there exists a unique solution in the domain of the generator to problem (2.1).

(ii) Assume (H1a) and (H1b). If $g \in B([0,T] \times \Omega)$ and $\phi_0 \in Dom(\mathcal{L}_\Omega)$, then there exists a unique generalised solution to problem (2.1), and the generalised solution allows the stochastic representation for every $(t,x) \in (0,T] \times \Omega$

$$u(t,x) = \mathbf{E}\left[\phi(0, X^{x,\Omega}(\tau_0(t)))\mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}}\right] + \mathbf{E}\left[\int_0^{\tau_0(t)\wedge\tau_\Omega(x)} g\left(-X^{t,(\nu)}(s), X^{x,\Omega}(s)\right) ds\right]. \tag{2.7}$$

Proof. (i) Observe that the potential $(-\mathcal{L}^{\text{kill}}_{(\nu),\Omega})^{-1}$ maps $C_{0,\partial\Omega}([0,T]\times\Omega)$ to itself. This follows from $P_s^{(\nu),\Omega,\text{kill}} g \in C_{0,\partial\Omega}([0,T]\times\Omega)$ for $g \in C_{0,\partial\Omega}([0,T]\times\Omega)$, $s \geq 0$, and Dominated Convergence Theorem (DCT) with dominating function $G(s) := \|g\|_\infty \mathbf{P}[s < \tau_0(T)]$. Note that we use the first identity in (1.9) to prove that $G \in L^1(0,\infty)$. The potential $(-\mathcal{L}^{\text{kill}}_{(\nu),\Omega})^{-1}$ is also bounded by the inequality

$$\left|(-\mathcal{L}^{\text{kill}}_{(\nu),\Omega})^{-1} g(t,x)\right| \leq \|g\|_\infty \mathbf{E}\left[\tau_0(T)\right], \quad g \in C_{0,\partial\Omega}([0,T]\times\Omega).$$

It then follows by [Dynkin, 1965, Theorem 1.1'] that $\bar{u} := (-\mathcal{L}^{\text{kill}}_{(\nu),\Omega})^{-1}(f + \mathcal{L}_\Omega\phi_0)$ is the unique solution to the abstract evolution equation

$$\mathcal{L}^{\text{kill}}_{(\nu),\Omega}\bar{u} = -(f + \mathcal{L}_\Omega\phi_0) \text{ on } (0,T]\times\overline{\Omega}, \quad \bar{u} = 0 \text{ on } \{0\}\times\overline{\Omega}, \quad \text{and } \bar{u} \in \text{Dom}(\mathcal{L}^{\text{kill}}_{(\nu),\Omega}). \tag{2.8}$$

It is now enough to show that $\bar{u}$ satisfies (2.8) if and only if $u = \bar{u} + \phi_0$ satisfies (4.8). For the 'if' direction, let $u \in \text{Dom}(\mathcal{L}_{(\nu),\Omega})$ satisfy (4.8). Note that $u(0) = \phi_0$. Then $\bar{u} := u - \phi_0 \in \text{Dom}(\mathcal{L}^{\text{kill}}_{(\nu),\Omega})$, and $\mathcal{L}_{(\nu),\Omega}\bar{u} = \mathcal{L}^{\text{kill}}_{(\nu),\Omega}\bar{u}$, by Theorem 1.6.3-(iv). So we can compute

$$\mathcal{L}^{\text{kill}}_{(\nu),\Omega}\bar{u} = \mathcal{L}_{(\nu),\Omega}(u - \phi_0) = \mathcal{L}_{(\nu),\Omega}u - \mathcal{L}_\Omega\phi_0 = -f - \mathcal{L}_\Omega\phi_0,$$

where we use

$$\mathcal{L}_{(\nu),\Omega}1\phi_0 = (\mathcal{L}_{(\nu)} + \mathcal{L}_\Omega)1\phi_0 = \mathcal{L}_\Omega\phi_0,$$

from Theorem 1.6.3-(i) taking the invariant cores $\mathcal{C}_{(\nu)} = \text{Dom}(\mathcal{L}_{(\nu)})$ and $\mathcal{C}_\Omega = \text{Dom}(\mathcal{L}_\Omega)$ (recalling that $\mathcal{L}_{(\nu)}1 = 0$). For the 'only if' direction, let $\bar{u}$ satisfy (2.8), and define $u := \bar{u} + \phi_0$. Then with the same justifications as just above, compute

$$\mathcal{L}_{(\nu),\Omega}u = \mathcal{L}^{\text{kill}}_{(\nu),\Omega}\bar{u} + \mathcal{L}_{(\nu),\Omega}\phi_0 = -(f + \mathcal{L}_\Omega\phi_0) + \mathcal{L}_\Omega\phi_0 = -f.$$

It follows that

$$u = (-\mathcal{L}^{\text{kill}}_{(\nu),\Omega})^{-1}(f + \mathcal{L}_\Omega \phi_0) + \phi_0.$$

(ii) Let $f \in B([0,T] \times \bar{\Omega})$. Then $f + \mathcal{L}_\Omega \phi_0 \in B([0,T] \times \bar{\Omega})$. Now take a sequence $\{\tilde{f}_n\}_{n\mathbb{N}} \in C_{0,\partial\Omega}([0,T] \times \Omega)$ such that $\tilde{f}_n \to f + \mathcal{L}_\Omega \phi_0$ a.e., and $\sup_n \|\tilde{f}_n\|_\infty < \infty$. Define $f_n := \tilde{f}_n - \mathcal{L}_\Omega \phi_0$ for each $n \in \mathbb{N}$ and note that $f_n \to f$ a.e., $\sup_n \|\tilde{f}_n\|_\infty < \infty$ and $f_n(0) = -\mathcal{L}_\Omega \phi_0$, as required by Definition 2.1.2. Now, for each f_n consider the stochastic representation of the respective solution in the domain of the generator

$$u_n(t,x) = \mathbf{E}\left[\int_0^{\tau_{t,x}} f_n\left(-X^{t,(\nu)}(s), X^{x,\Omega}(s)\right) ds\right] + \mathbf{E}\left[\int_0^{\tau_{t,x}} \mathcal{L}_\Omega \phi_0\left(X^{x,\Omega}(s)\right) ds\right] + \phi_0(x).$$

Fix $(t,x) \in (0,T] \times \Omega$. Using absolute continuity with respect of Lebesgue measure of the laws of $-X^{t,(\nu)}(s)$ and $X^{x,\Omega}(s)$ for each $s > 0$, and the bound $\mathbf{E}[\tau_{t,x}] \le \mathbf{E}[\tau_0(t)] < \infty$, we can apply DCT twice to obtain as $n \to \infty$

$$\begin{aligned}
\mathbf{E}\left[\int_0^{\tau_{t,x}} f_n\left(-X^{t,(\nu)}(s), X^{x,\Omega}(s)\right) ds\right] &= \int_0^\infty P_s^{(\nu),\text{kill}} P_s^\Omega f_n(t,x)\, ds \\
&\to \int_0^\infty P_s^{(\nu),\text{kill}} P_s^\Omega f(t,x)\, ds \\
&= \mathbf{E}\left[\int_0^{\tau_{t,x}} f\left(-X^{t,(\nu)}(s), X^{x,\Omega}(s)\right) ds\right],
\end{aligned}$$

using as a dominating function $G := \sup_n \|f_n\|_\infty$ to show that for each $s > 0$

$$F_n(s) := P_s^{(\nu),\text{kill}} P_s^\Omega f_n(t,x) \to P_s^{(\nu),\text{kill}} P_s^\Omega f(t,x) =: F(s),$$

and the dominating function $G(s) := \sup_n \|f_n\|_\infty \mathbf{P}[s < \tau_{t,x}]$ to show that

$$\int_0^\infty F_n(s)\, ds \to \int_0^\infty F(s)\, ds.$$

The convergence on $[0,T] \times \partial\Omega \cup \{0\} \times \bar{\Omega}$ is trivial. It follows that a generalised solution u exists and it is given by

$$u = (-\mathcal{L}^{\text{kill}}_{(\nu),\Omega})^{-1}(f + \mathcal{L}_\Omega \phi_0) + \phi_0.$$

Finally, independence of the approximating sequence proves uniqueness.

(iii) This is a standard application of Dynkin formula ([Dynkin, 1965, Theorem 5.1]) using the integrable stopping times $\tau_{t,x}$, $(t,x) \in (0,T] \times \Omega$, namely

$$(-\mathcal{L}^{\text{kill}}_{(\nu),\Omega})^{-1}(\mathcal{L}_\Omega \phi_0)(t,x) = \mathbf{E}\left[\int_0^{\tau_{t,x}} \mathcal{L}_{(\nu),\Omega} \phi_0\left(X^{x,\Omega}(s)\right) ds\right] = \mathbf{E}\left[\phi_0(X^{x,\Omega}(\tau_{t,x}))\right] - \phi_0(x).$$

□

Example 2.1.5. In the Caputo case, i.e. $\nu(t,r) = \nu(r) = -r^{-1-\beta}/\Gamma(-\beta)$, $\beta \in (0,1)$, the generalised solution (2.7) reads

$$\frac{t}{\beta}\int_0^\infty P_s^\Omega \phi_0(s) s^{-\frac{1}{\beta}-1} p_1^\beta\left(ts^{-\frac{1}{\beta}}\right) ds + \int_0^\infty \int_0^t P_s^\Omega g(r,x) p_s^\beta(t-r)\, dr\, ds,$$

using the independence of the processes X^Ω and τ_0, and X^Ω and X^β, and the know formula for the law of τ_0 in the first term, using the notation in Section 1.4.3. The homogeneous term is indeed the known solution to the Caputo EE (see, e.g., Meerschaert and Scheffler [2004]).

Remark 2.1.6. We could weaken the regularity of ϕ_0 by another a.e. approximation by an appropriate bounded sequence.

Remark 2.1.7. If assumption (H1) does not hold, one can modify the definition of a generalised solution requiring pointwise convergence everywhere on $(0,T]\times\Omega$ of the approximating sequence. This allows to run the argument of Theorem 2.1.4-(ii) as long as one such sequence exists. This means that our data g has to be a Baire class 1 function (which includes continuous functions but it is a smaller class than $B([0,T]\times\overline{\Omega})$).

2.2 Series Approximation

2.2.1 Generalised RL integral operator $I_0^{(\nu)}$

We use the potential operator corresponding to the generator $-D_0^{(\nu)}$ as in Definition 1.3.10-(iii) to define an integral operator on $B[0,T]$, which can be thought of as a generalisation of the RL integral operator I_0^β of order $\beta\in(0,1)$ (see, e.g., [Diethelm, 2010, Definition 2.1]).

Definition 2.2.1. Let ν be a function satisfying assumption (H0). The operator $I_0^{(\nu)} : B[0,T]\to B[0,T]$ defined by

$$\left(I_0^{(\nu)} f\right)(t) := \int_{(0,t]} f(y)\left(\int_0^\infty p_s^{(\nu)}(t,dy)ds\right), \quad t>a,$$

and 0 for $t=0$, will be called *the generalised RL fractional integral* associated with ν.

The generalised fractional integral $I_0^{(\nu)}$ satisfies the following:

(i) for the process $-X^{t,(\nu)}$ we have

$$I_0^{(\nu)} f(t) = \mathbf{E}\left[\int_0^{\tau_0(t)} f(-X^{t,(\nu)}(s))\, ds\right],$$

which follows from Proposition 1.3.11-(i)-(ii).

(ii) For each $f\in B[0,T]$,

$$\left|\left(I_0^{(\nu)} f\right)(t)\right| \le \|f\| \sup_{t\in[0,T]} \mathbf{E}\left[\tau_0(t)\right].$$

In particular, if $f = \mathbf{1}$ (the constant function 1), then

$$\left(I_0^{(\nu)}\mathbf{1}\right)(t) = \int_{(0,t]}\int_0^\infty p_s^{(\nu)}(t,dy)\,ds = \mathbf{E}\left[\tau_0(t)\right].$$

Remark 2.2.2. The operator $I_0^{(\nu)}$ is the left inverse operator of the RL-type operator $(\mathcal{L}_{(\nu)}^{\mathrm{kill}}, \mathrm{Dom}(\mathcal{L}_{(\nu)}^{\mathrm{kill}}))$.

Remark 2.2.3. If $\nu(t,y) = -y^{-1-\beta}/\Gamma(-\beta)$, $\beta \in (0,1)$, then $I_0^{(\nu)}$ coincides with the Riemann-Liouville integral operator I_0^β of order β (see, e.g., [Diethelm, 2010, Chapter 2]). Let $\tau_0^\beta(t)$ be the first exit time from the interval $(0,T]$ of $t - X^\beta$, where X^β is the β-stable subordinator. If $p_s^\beta(t,y)$ denotes the transition density of $t + X^\beta(s)$, it is well known that for for $y > t$

$$p_s^\beta(t,y) = s^{-1/\beta}p_1^\beta(s^{-1/\beta}(y-t)).$$

Then $p_s^\beta(t-y)$, $t > y$, is the transition density of the respective inverted β-stable subordinator started at t. Then for $y < t$

$$\begin{aligned}\int_0^\infty p_s^\beta(t-y)\,ds &= \int_0^\infty s^{-1/\beta}p_1^\beta(s^{-1/\beta}(t-y))\,ds\\ &= (t-y)^{\beta-1}\int_0^\infty u^{-1/\beta}p_1^\beta(u^{-1/\beta})du = \frac{1}{\Gamma(\beta)}(t-y)^{\beta-1},\end{aligned} \tag{2.9}$$

using the Mellin transform of the β-stable density p_1^β for the last equality (see, e.g., [Zolotarev, 1986, Theorem 2.6.3, p. 117]). The previous yields the known results

$$\left|\left(I_0^\beta f\right)(t)\right| \le \frac{1}{\Gamma(\beta+1)}\|f\|_\infty T^\beta,$$

and

$$\left(I_0^\beta\mathbf{1}\right)(t) = \int_0^t\int_0^\infty p_s^\beta(t-y)\,ds\,dy = \mathbf{E}\left[\tau_0^\beta(t)\right] = \frac{t^\beta}{\Gamma(\beta+1)}.$$

Let $I_0^{(\nu),n}$ denote the n-fold iteration of the operator $I_0^{(\nu)}$, $n \in \mathbb{N}$. For convention $I_0^{(\nu),0}$ stands for the identity operator. We denote by $B(\gamma,\rho) = \int_0^1 s^{\gamma-1}(1-s)^{\rho-1}\,ds$, $\gamma,\rho > 0$ the standard Beta function.
The following result is key for this section, as it provides an explicit bound for $|I_0^{(\nu),n}f|$ under assumption (H2).

Theorem 2.2.4. *Let ν be a function satisfying assumptions (H0), (H2). Then, for each $f \in B[0,T]$,*

$$\left|\left(I_0^{(\nu),n}f\right)(t)\right| \le \|f\|_t\frac{T^{n\beta}}{(\Gamma(\beta+1))^n}\prod_{k=0}^{n-1}B(k\beta+1,\,\beta),\quad n \ge 1, \tag{2.10}$$

where $\|f\|_t := \sup_{y\le t}|f(y)|$. Moreover, the series

$$\sum_{n=0}^\infty\left(I_0^{(\nu),n}f\right)(t) \tag{2.11}$$

converges uniformly on $[0,T]$.

Proof. By definition of the generalised fractional integral

$$\begin{aligned}\left|\left(I_{0*}^{(\nu)}f\right)(t)\right| &\leq \int_0^\infty \left(\int_{(0,t]} |f(y)|\, p_s^{(\nu)}(t,dy)\right) ds \\ &\leq \int_0^\infty \left(\int_{(0,t]} \sup_{z\leq y}|f(z)| p_s^{(\nu)}(t,dy)\right) ds.\end{aligned}$$

Fix $\beta \in (0,1)$ as in (H2) and denote by $\{t - X^\beta(s)\}_{s\geq 0}$ the associated inverted β-stable subordinator. By assumption (H2) it follows from [Zhang, 2007, Theorem 1.5] that $\mathbf{P}[-X^{t,(\nu)}(s) > y] \leq \mathbf{P}[-X^{t,\beta}(s) > y],\ t,y \in (0,T],\ s\in(0,\infty)$. Therefore

$$\mathbf{E}\left[g\left(-X_0^{t,(\nu)}(s)\right)\right] = \mathbf{E}\left[g\left(-X^{t,(\nu)}(s)\right)\right] \leq \mathbf{E}\left[g\left(t - X^\beta(s)\right)\right]$$

for any non-decreasing function $g \in C_\infty^1(-\infty,b]$ such that $g(t) = 0,$ for all $x \leq a$, where the equality holds as a consequence of the proof of Proposition 1.3.11-(i). By a standard approximation argument we obtain

$$\mathbf{P}[-X_0^{t,(\nu)}(s) > y] \leq \mathbf{P}[-X^{t,\beta}(s) > y], \quad t,y\in(0,T],\ s\in(0,\infty).$$

Another approximation argument yields

$$\mathbf{E}\left[g\left(-X_0^{t,(\nu)}(s)\right)\right] \leq \mathbf{E}\left[g\left(t - X^\beta(s)\right)\right], \tag{2.12}$$

for any non-decreasing bounded function $g : [0,T] \to \mathbb{R}$. In particular (2.12) holds for the function $g(y) = \sup_{z\leq y}|f(z)|$. Hence

$$\begin{aligned}\left|\left(I_0^{(\nu)}f\right)(t)\right| &\leq \int_0^\infty \left(\int_{(0,t]} |f(y)| p_s^{(\nu)}(t,dy)\right) ds \\ &\leq \int_0^\infty \int_0^t \sup_{z\leq y}|f(z)| p_s^\beta(t-y)\,dy\,ds \\ &\leq \|f\|_t \int_0^\infty \int_0^t p_s^\beta(t-y)\,dy\,ds \leq \frac{1}{\Gamma(\beta+1)}\|f\|_t t^\beta, \end{aligned}\tag{2.13}$$

To prove the inequality (2.10) we proceed by induction. Case $n=1$ is given by (2.13). Assume that the inequality in (2.10) holds for $n-1$. Then, using standard identities for the Beta function, the inequality in (2.13) and the induction hypothesis

$$\begin{aligned}
\left| \left(I_0^{(\nu),n} f \right)(t) \right| &= \left| I_0^{(\nu)} I_0^{(\nu),n-1} f(t) \right| \le \int_0^\infty \int_0^t \sup_{z \le y} \left| I_0^{(\nu),n-1} f(z) \right| p_s^\beta(t-y)\, dy\, ds \\
&\le \int_0^\infty \int_0^t \|f\|_y \frac{y^{(n-1)\beta}}{(\Gamma(\beta+1))^{n-1}} \prod_{k=0}^{n-2} B(k\beta+1,\, \beta) p_s^\beta(t-y)\, dy\, ds \\
&\le \|f\|_t \frac{1}{(\Gamma(\beta+1))^{n-1}} \prod_{k=0}^{n-2} B(k\beta+1,\, \beta) \int_0^\infty \int_a^t y^{(n-1)\beta} p_s^\beta(t-y)\, dy\, ds \\
&\le \|f\|_t \frac{1}{(\Gamma(\beta+1))^{n-1}} \prod_{k=0}^{n-2} B(k\beta+1,\, \beta) \int_0^t y^{(n-1)\beta}(t-y)^{\beta-1} \frac{1}{\Gamma(\beta+1)}\, dy \\
&= \|f\|_t \frac{T^{n\beta}}{(\Gamma(\beta+1))^{n}} \prod_{k=0}^{n-1} B(k\beta+1,\, \beta),
\end{aligned}$$

where the last inequality uses Fubini's Theorem and the equality in (2.9).
To prove the convergence of (2.11) we use the known identities

$$\Gamma(z+1) = z\Gamma(z), \qquad B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{2.14}$$

and inequality

$$\Gamma(na) \ge (n-1)! a^{2(n-1)} \big(\Gamma(a)\big)^n, \tag{2.15}$$

for $n \in \mathbb{N}$ and $a > 0$, to obtain that for each $n \in \mathbb{N}$

$$\prod_{k=0}^{n-1} B(k\beta+1,\beta) = \frac{(\Gamma(\beta)\ )^n}{n\beta\Gamma(n\beta)} \le \frac{(\ \Gamma(\beta)\ )^n}{n\beta(n-1)!\beta^{2(n-1)}\,(\ \Gamma(\beta)\ )^n} \le \frac{1}{n!\beta^{2n}}.$$

Hence,

$$\left| \left(I_0^{(\nu),n} f \right)(t) \right| \le \|f\|_\infty \left(\frac{(b-a)^\beta}{\beta^2\Gamma(\beta+1)} \right)^n \frac{1}{n!} =: M_n.$$

Since $\sum_{n=0}^\infty M_n$ converges, Weierstrass M-test implies the uniform convergence of (2.11) on $[0,T]$, as required.

□

Remark 2.2.5. In the classical fractional setting, the n-fold RL integral $I_0^{\beta,n}$ has an explicit expression obtained from its semigroup property [Diethelm, 2010, Theorem 2.2]

$$\left(I_0^{\beta,n} f \right)(t) = \left(I_0^{n\beta} f \right)(t).$$

Hence, for $f(t) = \mathbf{1}$,

$$\left(I_0^{\beta,n} f \right)(t) = \frac{1}{\Gamma(n\beta)} \int_a^t (t-y)^{n\beta-1} dy = \frac{(t-a)^{n\beta}}{\Gamma(n\beta+1)}.$$

2.2.2 Series approximation of generalised solutions

We first construct a series representation of the generalised solution for $\mathcal{L}_\Omega$ bounded. Then, for a possibly unbounded spatial operator $\mathcal{L}_\Omega$, we show the convergence of the series corresponding to the Yosida approximations the to the generalised solution of $\mathcal{L}_\Omega$ (see Theorem 2.2.15 and Theorem 2.2.16 below).

Series representation for $\mathcal{L}_\Omega$ bouned

Under the additional assumptions

$$\mathcal{L}_\Omega \text{ is bounded and } \nu \text{ satisfies assumption (H2)},$$

we give a series representation for the generalised solution obtained in Theorem 2.1.4.
Let us give wellposedness and stochastic representation for the solution to the (FODE) problem

$$\mathcal{L}_{(\nu)} = -g, \text{ in } (0,T], \quad \text{and } u(0) = 0. \tag{2.16}$$

Definition 2.2.6. Let $g \in C_0[0,T]$. A function $u \in C_0[0,T]$ is a *solution in the domain of the generator to problem* (2.16) if $u \in \mathrm{Dom}(\mathcal{L}^{\mathrm{kill}}_{(\nu)})$ and u satisfies (2.16).

Definition 2.2.7. A function $u \in B[0,T]$ is a generalised solution to problem (2.16) if $u = \lim_{n\to\infty} u_n$ point-wise, where u_n is the solution in the domain of the generator to problem (2.16) for $g_n \in C_0[0,T]$, $n \in \mathbb{N}$, $g_n \to g$ a.e. and $\sup_{n\in\mathbb{N}} \|g_n\|_\infty < \infty$.

The following is just a simpler version of Theorem 2.1.4.

Theorem 2.2.8. *Let ν be a function satisfying conditions (H0), (H1a). If $g \in C_0[0,T]$, then there exists a unique solution in the domain of the generator $u \in C_0[0,T]$ to problem (2.16), and u has the representation $u = I_0^{(\nu)} g$.*
Under the additional assumption (H1a), if $g \in B[0,T]$ there exists a unique $u \in B[0,T]$ generalised solution to problem (2.16), also with the representation $u = I_0^{(\nu)} g$.

Theorem 2.2.9. *Let ν be a function satisfying assumption (H0), (H2). Suppose that $\mathcal{L}_\Omega$ is bounded.*

1. *If $g \in C_{0,\partial\Omega}([0,T]\times\Omega)$, then the unique solution in the domain of the generator to problem (2.1) has the series representation*

$$u(t,x) = \sum_{n=0}^{\infty} \left((I_0^{(\nu)} \mathcal{L}_\Omega)^n I_0^{(\nu)} g \right)(t,x), \tag{2.17}$$

where the convergence is in the sense of the norm of $C_{0,\partial\Omega}([0,T]\times\Omega)$.

(ii) *If $g \in B([0,T]\times\overline{\Omega})$ and (H1a) holds, the unique generalised solution $u \in B([0,T]\times\Omega)$ to problem (2.1) has the series representation given in (2.17).*

Proof. Note that by Riesz-Representation Theorem ([Kolokoltsov, 2011, Theorem 1.7.3]) $\mathcal{L}_\Omega$ and $I_0^{(\nu)}$ commute.

(i) Let $u \in C_{0,\partial\Omega}([0,T] \times \Omega)$ be the solution in the domain of the generator to problem (2.1) obtained in Theorem 2.1.4. As $\mathcal{L}_\Omega$ is bounded and $u \in \mathrm{Dom}(\mathcal{L}_{(\nu),\Omega})$ we obtain by Proposition 1.6.6 that for each $x \in \Omega$, $u(\cdot, x) \in \mathrm{Dom}(\mathcal{L}_{(\nu)})$, $Lu(\cdot, x) = (\mathcal{L}_{(\nu)} + \mathcal{L}_\Omega)u(\cdot, x)$. Hence $u(\cdot, x)$ solves

$$\mathcal{L}_{(\nu)}u(\cdot, x) = -\tilde{g}(\cdot, x), \quad u(0, x) = 0 \tag{2.18}$$

where $\tilde{g}(\cdot, x) := \mathcal{L}_\Omega u(\cdot, x) + g(\cdot, x) \in C_0[0, T]$, as $\mathcal{L}_\Omega u(0) = 0$. Hence, by Theorem 2.2.8, $u(\cdot, x)$ is the unique solution in the domain of the generator to problem (2.18) and it has the representation $u(\cdot, x) = I_0^{(\nu)}\tilde{g}(\cdot, x)$.

By induction, for each $N \in \mathbb{N}$

$$u(t, x) = \sum_{n=0}^{N} \left((I_0^{(\nu)}\mathcal{L}_\Omega)^n I_0^{(\nu)} g \right)(t, x) + \left((I_0^{(\nu)}\mathcal{L}_\Omega)^{N+1} u \right)(t, x). \tag{2.19}$$

Now observe that,

$$\begin{aligned} a_n(t, x) := \left((I_0^{(\nu)}\mathcal{L}_\Omega)^n I_0^{(\nu)} g \right)(t, x) &\le \left| \left((I_0^{(\nu)}\mathcal{L}_\Omega)^n I_0^{(\nu)} g \right)(t, x) \right| \\ &\le \|g\|_\infty \|\mathcal{L}_\Omega\|^n \left| \left(I_0^{(\nu),n+1} \mathbf{1} \right)(t) \right| =: b_n(t). \end{aligned}$$

Hence Theorem 2.2.4 implies the uniform convergence of $\sum_{n=0}^{\infty} b_n(t)$, which in turn implies the uniform convergence of $\sum_{n=0}^{\infty} a_n(t, x)$. Moreover

$$\left| \left((I_0^{(\nu)}\mathcal{L}_\Omega)^{N+1} u \right)(t, x) \right| \le \|u\|_\infty \|\mathcal{L}_\Omega\|^N \left| I_0^{(\nu),N+1}(t, x) \right| \to 0, \quad N \to \infty,$$

due to the uniform convergence of $\sum_{n=0}^{\infty} \|\mathcal{L}_\Omega\|^n \left(I_0^{(\nu),n} \mathbf{1} \right)(t)$ on $[0, T]$, again by Theorem 2.2.4. Then, letting $N \to \infty$ in the equality (2.19) yields the result in (2.17).

(ii) Consider a sequence $\{g_n\}_{n \in \mathbb{N}} \subset C_{0,\partial\Omega}([0, T] \times \Omega)$ such that $g_n \to g$ a.e. and $\sup_n \|g_n\|_\infty < \infty$. Fix $(t, x) \in [0, T] \times \Omega$. By DCT we obtain

$$\lim_{n \to \infty} \sum_{m=0}^{\infty} F_{(t,x),n}(m) = \sum_{m=0}^{\infty} \left((I_0^{(\nu)}\mathcal{L}_\Omega)^m I_0^{(\nu)} g \right)(t, x), \tag{2.20}$$

where $F_{(t,x),n}(m) := (I_0^{(\nu)}\mathcal{L}_\Omega)^m I_0^{(\nu)} g_n$. To see this observe that for every $m \in \mathbb{N}$

$$\lim_{n \to \infty} F_{(t,x),n}(m) = \left((I_0^{(\nu)}\mathcal{L}_\Omega)^m I_0^{(\nu)} g \right)(t, x),$$

and $|F_{(t,x),n}(m)| \le F_{(t,x)}(m) := \sup_n \|g_n\|_\infty \|\mathcal{L}_\Omega\|^m (I_0^{(\nu)})^{m+1}(1)(t)$.

By part (i) of this Theorem and part (ii) of Theorem 2.1.4 the limit on the left-hand-side of (2.20) equals the unique generalised solution to problem (2.1).

$\square$

Theorem 2.2.10. *Let ν be a function satisfying conditions (H0), (H2). Let $\mathcal{L}_\Omega$ be a bounded linear operator on $C_{\partial\Omega}(\Omega)$.*

(i) If $g \in C_{\partial\Omega}([0,T]\times\Omega)$, $\phi_0 \in Dom(\mathcal{L}_\Omega)$, $\mathcal{L}_\Omega\phi_0(\cdot) = -g(a,\cdot)$, then the unique solution $u \in C_{\partial\Omega}([0,T]\times\Omega)$ in the domain of the generator to problem (2.1) has the series representation

$$u(t,x) = \sum_{n=0}^{\infty} \mathcal{L}_\Omega^n \phi_0 I_0^{(\nu),n} 1(t,x) + \sum_{n=0}^{\infty} (I_0^{(\nu)}\mathcal{L}_\Omega)^n I_0^{(\nu)} g(t,x). \tag{2.21}$$

(ii) If $g \in B([0,T]\times\overline{\Omega})$, $\phi_0 \in Dom(\mathcal{L}_\Omega)$, condition (H1a) holds, then the unique generalised solution $u \in B([0,T]\times\overline{\Omega})$ to problem (2.1) has the series representation (2.21).

Proof.

(i) Let $u \in C_{\partial\Omega}([0,T]\times\Omega)$ be the solution in the domain of the generator to problem (2.1). By Proposition 1.6.6, $\tilde{u} := u - \phi_0 \in \mathrm{Dom}(\mathcal{L}_\Omega) \subset C_{0,\partial\Omega}([0,T]\times\Omega)$ solves

$$-\mathcal{L}_{(\nu)}\tilde{u}(t,x) = -\mathcal{L}_\Omega\tilde{u}(t,x) - (g(t,x) + \mathcal{L}_\Omega\phi_0(x)), \quad \tilde{u}(a,\cdot) = 0. \tag{2.22}$$

By the assumptions of the Theorem $\tilde{g} := g + \mathcal{L}_\Omega\phi_0 \in C_{0,\partial\Omega}([0,T]\times\Omega)$. Therefore by Theorem 2.2.9-(i) $\tilde{u}$ is the unique solution in the domain of the generator to problem (2.22) and it has the series representation

$$\begin{aligned}\tilde{u}(t,x) &= \sum_{n=0}^{\infty}(I_0^{(\nu)}\mathcal{L}_\Omega)^n I_0^{(\nu)}\tilde{g}(t,x)\\ &= \sum_{n=0}^{\infty}(I_0^{(\nu)}\mathcal{L}_\Omega)^n I_0^{(\nu)} g(t,x) + \sum_{n=0}^{\infty}(I_0^{(\nu)}\mathcal{L}_\Omega)^n I_0^{(\nu)}\mathcal{L}_\Omega\phi_0(t,x).\end{aligned} \tag{2.23}$$

using the fact that both series in the right-hand side converge in $C_{\partial\Omega}([0,T]\times\Omega)$ by Theorem 2.2.4. Then $u = \tilde{u} + \phi_0$ has the series representation given in (2.21).

(ii) For $g \in B([0,T]\times\Omega)$, let $\tilde{u}$ be the unique generalised solution to problem (2.1) with $\tilde{g} = g + \mathcal{L}_\Omega\phi_0$. Then by Theorem 2.2.9-(ii) $\tilde{u}$ has the representation (2.23), using the fact that both series in the right-hand side converge in $B([0,T]\times\Omega)$ by Theorem 2.2.4. Then $u = \tilde{u} + \phi_0$ has representation (2.21).

□

Generalised Mittag-Leffler operators

Remark 2.2.11. Theorem 2.2.16 allows us to give meaning to a generalised Mittag-Leffler function for $\mathcal{L}_\Omega$ generator of a Feller semigroup on $C_{\partial\Omega}(\Omega)$.

Definition 2.2.12. Let ν satisfy conditions (H0), (H2) and let $\mathcal{L}_\Omega$ be bounded. We call $E_{(\nu)}(\mathcal{L}_\Omega(\cdot)I_0^{(\nu)}1) : B(\Omega) \to B([0,T]\times\Omega)$ *the generalised Mittag-Leffler function for $\mathcal{L}_\Omega$ and ν*, defined as

$$\phi_0 \mapsto E_{(\nu)}(\mathcal{L}_\Omega\phi_0 I_0^{(\nu)}1)(t,x) := \sum_{n=0}^{\infty}\mathcal{L}_\Omega^n\phi_0(x) I_0^{(\nu),n}1(t), \tag{2.24}$$

$(t,x) \in [0,T]\times\Omega$.

Remark 2.2.13. The function $E_{(\nu)}(\mathcal{L}_\Omega(\cdot)I_0^{(\nu)}1)$ provides a probabilistic generalisation, for $\lambda = \mathcal{L}_\Omega$ bounded operator, to the Mittag-Leffler function

$$E_\beta(\lambda t^\beta) = \sum_{n=0}^{\infty} \frac{\lambda^n t^{\beta n}}{\Gamma(\beta n+1)} = \sum_{n=0}^{\infty} \lambda^n \phi_0(x) I_0^{\beta,n}(1)(t),$$

where $\beta \in (0,1)$, $\phi_0(\cdot) = 1$.

Convergence of the series representation to the stochastic representation

We use the following lemma to exploit Yosida operators os our approximating bounded operators.

Lemma 2.2.14. Let $\mathcal{L}_{\Omega,\lambda} := \lambda\mathcal{L}_\Omega(\lambda - L_\Omega)^{-1}$ be the Yosida approximation for the generator $\mathcal{L}_\Omega$ of a Feller semigroup on $C_{\partial\Omega}(\Omega)$. Let $g \in C_{\partial\Omega}([0,T]\times\Omega)$. Let $u_\lambda \in C_{0,\partial\Omega}([0,T]\times\Omega)$ be the generalised solution to problem (2.1) with $\mathcal{L}_\Omega \equiv L_{\Omega,\lambda}$. Let $u \in C_{0,\partial\Omega}([0,T]\times\Omega)$ be the generalised solution to problem (2.1), with $\mathcal{L}_\Omega = L$.
Then for each $t \in [0,T]$, $u_\lambda(t,x) \to u(t,x)$ as $\lambda \to \infty$, uniformly in $x \in \Omega$.

Proof. By [Ethier and Kurtz, 2009, Chapter 1, Proposition 2.7] we have that for each $g \in C_{\partial\Omega}([0,T]\times\Omega)$, $t \in [0,T]$,

$$\|(P_s^{\Omega,\lambda} - P_s^{\Omega,\lambda})g(t,\cdot)\|_{C(\bar{\Omega})} \to 0 \quad \text{as } \lambda \to \infty,$$

uniformly for $s \geq 0$ in compact sets. Pick the constant function $\|g\|_\infty$ as the dominating function. Then $\|P_s^{\Omega,\lambda} g(t,\cdot)(x)\| \leq 1\|g(\cdot,x)\| \leq \|g\|_\infty$ which implies

$$\mathbf{E}\left[\int_0^{\tau_0(t)} |P_s^{\Omega,\lambda} g(-X^{t,(\nu)}(s),\cdot)(x)| ds\right] \leq \|g\|_\infty E\left[\tau_0(t)\right] < \infty,$$

and the result follows from the application of DCT. $\square$

Theorem 2.2.15. *Let ν be a function satisfying assumptions (H0), (H2). Let $\mathcal{L}_{\Omega,\lambda}$ be the Yosida approximation for the generator of a Feller semigroup $\mathcal{L}_\Omega$ on $C_{\partial\Omega}(\Omega)$, such that (H1b) holds. Let $g \in C_{\partial\Omega}([0,T]\times\Omega)$.*
Then for each $t \in [0,T]$, as $\lambda \to \infty$

$$\sum_{n=0}^{\infty} (I_0^{(\nu)}\mathcal{L}_{\Omega,\lambda})^n I_0^{(\nu)} g(t,x) \to \mathbf{E}\left[\int_0^{\tau_0(t)\wedge\tau_\Omega(x)} P_s^\Omega g(-X^{t,(\nu)}(s),\cdot)(x)\, ds\right], \tag{2.25}$$

uniformly in $x \in \Omega$.

Proof. The result follows from combining Lemma 2.2.14 with Theorem 2.2.9. $\square$

Theorem 2.2.16. *Let ν be a function satisfying (H0), (H2), and assume that (H1a) holds. Let $\mathcal{L}_\Omega$ be the generator of a Feller semigroup on $C_{\partial\Omega}(\Omega)$, such that (H1b) holds, and let $\mathcal{L}_{\Omega,\lambda}$ its Yosida approximation.*
Fix $g \in C_{\partial\Omega}([0,T]\times\Omega)$ and $\phi_0 \in Dom(\mathcal{L}_\Omega)$.

Then for each $t \geq 0$

$$E_{(\nu)}(\mathcal{L}_\Omega \phi_0 I_0^{(\nu)} 1)(t,x) \to \mathbf{E}\left[\phi_0(X^{x,\Omega}(\tau_0(t))\mathbf{1}_{\tau_0(t)<\tau_\Omega(x)})\right],$$

and

$$\sum_{n=0}^{\infty}(I_0^{(\nu)}\mathcal{L}_{\Omega,\lambda})^n I_0^{(\nu)} g(t,x) \to \mathbf{E}\left[\int_0^{\tau_0(t)} P_s^{\Omega} g(-X^{t,(\nu)}(s),\cdot)(x)\,ds\right]$$

as $\lambda \to \infty$, uniformly in $x \in \Omega$.

Proof. Let $u_\lambda \in B([0,T]\times\Omega)$ be the generalised solution for problem (2.1) for $\mathcal{L}_\Omega \equiv \mathcal{L}_{\Omega,\lambda}$. Let $u \in B([0,T]\times\Omega)$ be the generalised solution to problem (2.1) for $\mathcal{L}_\Omega = \mathcal{L}_\Omega$ (with understanding of the ambiguity).
By Theorem 2.1.4

$$u_\lambda(t,x) = \mathbf{E}\left[\phi_0(X^{x,\Omega,\lambda}(\tau_0(t))) + \int_0^{\tau_0(t)} P_s^{\Omega,\lambda} g(-X^{t,(\nu)}(s),\cdot)(x)\,ds\right], \tag{2.26}$$

and

$$u(t,x) = \mathbf{E}\left[\phi_0(X^{x,\Omega}(\tau_0(t))) + \mathbf{E}\int_0^{\tau_0(t)} P_s^{\Omega}(-X^{t,(\nu)}(s),\cdot)(x)\,ds\right]. \tag{2.27}$$

As a consequence of Theorem 2.2.9 and Theorem 2.1.4 the second term in (2.26) equals the series representation (2.17) and by Theorem 2.2.15 it converges as required to the second term in (2.27).
The considerations above along with Theorem 2.2.10 imply that the first term in (2.26) equals the first term on the right-hand side of (2.21). For the first term in (2.26) observe that by [Ethier and Kurtz, 2009, Chapter 1, Proposition 2.7]

$$P_s^{\Omega,\lambda}\phi_0(x) \to P_s^{\Omega}\phi_0(x), \quad \lambda \to \infty,$$

uniformly in $x \in \Omega$, for each $s \geq 0$. For each $\lambda \geq 0$

$$\mathbf{E}\left[\phi_0\left(X^{x,\Omega,\lambda}(\tau_0(t))\right)\right] = \int_0^\infty P_s^{\Omega,\lambda}\phi_0(x) p^{\tau_0(t)}(ds),$$

by independence of $X^{x,\Omega,\lambda}$ and $\tau_0(t)$, where $p^{\tau_0(t)}(ds)$ is the law of $\tau_0(t)$. Also

$$|P_s^{\Omega,\lambda}\phi_0(x)| \leq \|\phi_0\|_\infty \quad \text{for all } \lambda > 0, \quad \text{and} \quad \int_0^\infty \|\phi_0\| p^{\tau_0(t)}(ds) \leq \|\phi_0\|_\infty,$$

and the result follows from the application of DCT. □

2.3 Wellposedness of the nonlinear Caputo-type EE

Let us now study the wellposedness for the nonlinear equation (2.2). We introduce a notion of solution and then we proceed as in Hernández-Hernández and Kolokoltsov [2016] via fixed point arguments.

Definition 2.3.1. Let ν be a function satisfying (H0), (H2). A function $u : [0,T] \times \Omega \to \mathbb{R}$ is said to be a generalised solution to the non-linear equation (2.2) if u is a generalised solution to the linear equation (2.1) with $g(t,x) := f(t,x,u(t,x))$ for all $(t,x) \in [0,T] \times \Omega$.

Lemma 2.3.2. Let ν be a function satisfying conditions (H0) and (H2). Assume that $\mathcal{L}_\Omega$ is the generator of a Feller semigroup on $C_{\partial\Omega}(\Omega)$ and $\phi_0 \in \mathrm{Dom}(\mathcal{L}_\Omega)$ and that (H1a) holds. Suppose that $f : [0,T]\times\Omega\times\mathbb{R} \to \mathbb{R}$ is a bounded measurable function. Then, a function $u \in C([0,T]\times\overline{\Omega})$ is a generalised solution to equation (2.2) if, and only if, u solves the non-linear integral equation

$$\begin{aligned} u(t,x) = &\int_0^\infty (P_s^\Omega \phi_0)(x) p^{\tau_0(t)}(ds) \\ &+ \mathbf{E}\left[\int_0^{\tau_0(t)} P_s^\Omega f\left(-X^{t,(\nu)}(s), \cdot, u(-X^{t,(\nu)}, \cdot)\right)(x)\, ds\right], \end{aligned} \tag{2.28}$$

where $p^{\tau_0(t)}$ is the law of $\tau_0(t)$.

Proof. By Definition 2.3.1, $u \in C([0,T] \times \overline{\Omega})$ is a generalised solution to (2.2) if and only if u is a generalised solution to the linear equation (2.1) with $g(t,x) := f(t,x,u(t,x))$. Note that if $u \in C([0,T] \times \overline{\Omega})$, then g is a measurable and bounded function on $[0,T] \times \Omega$. Hence Theorem 2.1.4-(ii) yields the integral equation (2.28), as required. □

Using Weissenger's fixed point theorem we prove that the integral equation (2.28) possesses a unique solution (for a given boundary ϕ_0) under the following additional assumption:

(H3) The function $f : [0,T]\times\Omega\times\mathbb{R} \to \mathbb{R}$ is bounded and fulfils the following Lipschitz condition with respect to the third variable: for all $(t,x,y_1),(t,x,y_2) \in [0,T] \times \Omega \times \mathbb{R}$,

$$|f(t,x,y_1) - f(t,x,y_2)| < L_f|y_1 - y_2|, \tag{2.29}$$

for a constant $L_f > 0$ (independent of t and x).

Theorem 2.3.3. *Let $[0,T] \subset \mathbb{R}$ and $\phi_0 \in Dom(\mathcal{L}_\Omega)$. Suppose that ν is a function satisfying conditions (H0), (H2). Suppose that (H1a) holds and that f is a function satisfying condition (H3). Then problem (2.2) has a unique generalised solution $u \in C([0,T] \times \Omega)$.*

Proof. By Lemma 2.3.2, the existence of a unique generalised solution to (2.2) means the existence of a unique solution to the integral equation (2.28). The latter equation can be rewritten as a fixed point problem $u(t,x) = (\Psi u)(t,x)$ for a suitable operator Ψ.

Step a) Definition of the operator Ψ. Denote by B_{ϕ_0} the closed convex subset of $C([0,T] \times \Omega)$ consisting of functions satisfying $f(0) = \phi_0$. This set is a metric space when endowed with the metric induced by the norm on $C([0,T] \times \Omega)$.
Next, define the operator Ψ on B_{ϕ_0} by

$$\begin{aligned} (\Psi u)(t,x) := &\int_0^\infty (P_s^\Omega \phi_0)(x) p^{\tau_0(t)}(ds) \\ &+ \mathbf{E}\left[\int_0^{\tau_0(t)} P_s^\Omega f\left(-X^{t,(\nu)}, \cdot, u(-X^{t,(\nu)}, \cdot)\right)(x)\, ds\right], \quad t \in [0,T]. \end{aligned} \tag{2.30}$$

Note that if $u \in B_{\phi_0}$, then $(\Psi u)(\cdot, x) \in C[0,T]$ for each $x \in \Omega$ and $(\Psi u)(t, \cdot) \in C(\Omega)$ for each $t \in [0,T]$. Further, $(\Psi u)(0,x) = \phi_0(x)$ as $\mu^{\tau_0(0)}(ds) = \delta_0(ds)$. Therefore, $\Psi : B_{\phi_0} \to B_{\phi_0}$.

Step b) Let Ψ^n denote the n-fold iteration of the operator Ψ for $n \geq 0$, $n \in \mathbb{N}$. For convention Ψ^0 denotes the identity operator. Note that for $n = 1$, the Lipschitz condition of f and the fact that P_s^Ω is a contraction semigroup imply

$$
\begin{aligned}
\Big|\Psi u - \Psi v\Big|(t,x) &= \left|\mathbf{E}\left[\int_0^{\tau_0(t)} P_s^\Omega \left(f\left(-X^{t,(\nu)}, \cdot, u(-X^{t,(\nu)}, \cdot)\right) - f\left(-X^{t,(\nu)}, \cdot, v(-X^{t,(\nu)}, \cdot)\right)\right)(x)\, ds\right]\right| \\
&\leq \mathbf{E}\left[\int_0^{\tau_0(t)} P_s^\Omega \left(\left|f\left(-X^{t,(\nu)}, \cdot, u(-X^{t,(\nu)}, \cdot)\right) - f\left(-X^{t,(\nu)}, \cdot, v(-X^{t,(\nu)}, \cdot)\right)\right|\right)(x)\, ds\right] \\
&\leq L_f \|u - v\|_t I_0^{(\nu)}(1)(t),
\end{aligned}
$$

where

$$
\|u - v\|_t := \sup_{z \leq t} \|u(z, \cdot) - v(z, \cdot)\|, \quad t \in [0,T],
$$

and L_f is the Lipschitz constant of the function f. Proceeding by induction we can prove that

$$
|\Psi^n u(t,x) - \Psi^n v(t,x)| \leq \|u - v\|_t L_f^n \left(I_0^{(\nu),n} \mathbf{1}\right)(t), \qquad n \geq 0,
$$

where $I_0^{(\nu),n}$ is the nth fold iteration of the generalised fractional operator $I_0^{(\nu)}$. Moreover, by Theorem 2.2.4, we know that

$$
\sum_{n=0}^{\infty} L_f^n \left(I_0^{(\nu),n} \mathbf{1}\right)(t) \leq \left(\frac{L_f^n (b-a)^\beta}{\beta^2 \Gamma(\beta+1)}\right)^n \frac{1}{n!} =: \alpha_n.
$$

Hence,

$$
\|\Psi^n u - \Psi^n v\| \leq \alpha_n \|u - v\|,
$$

for every $n \geq 0$ and every $u, v \in B_{\phi_0}$, where $\alpha_n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n$ converges.

Therefore, the Weissinger fixed point theorem [Diethelm, 2010, Theorem D.7] guarantees the existence of a unique fixed point $u^* \in B_{\phi_0}$ to the integral equation (2.28), which in turn implies the existence of a generalised solution to (2.2), as required. □

Chapter 3

Marchaud-type EE; stochastic weak solution

In this Chapter[1], we study the nonlocal-in-time evolution equation

$$\begin{cases} D_{\infty}^{(\nu)} u(t,x) - \Delta u(t,x) = f(t,x), & (t,x) \in (0,T] \times \Omega, \\ u(t,x) = 0, & (t,x) \in (0,T] \times \partial\Omega, \\ u(t,x) = \phi(t,x), & (t,x) \in (-\infty,0] \times \Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^d$ is a regular domain, the functions f and ϕ are given data, and $D_{\infty}^{(\nu)}$ denotes the nonlocal operator defined by

$$D_{\infty}^{(\nu)} u(t) := \int_0^{\infty} (u(t) - u(t-r))\nu(t,r)\, dr, \tag{3.2}$$

with the nonnegative kernel function $\nu \geq 0$ satisfying certain hypothesis. The nonlocal operator $-D_{\infty}^{(\nu)}$ is proved to be the Markovian generator of a $(-\infty, T]$-valued decreasing Lévy-type process, denoted by $-X^{t,(\nu)}$ when started at $t \in [0,T]$, as defined in Section 1.3.3. We denote by $B^x(s)$ a d-dimensional Brownian motion started at $x \in \mathbb{R}^d$ generated by the Laplacian Δ. The processes $-X^{t,(\nu)}$ and B^x are always assumed to be independent.

The aim of current chapter is to derive a stochastic representation for the solution to the problem (3.1) with the historical initial condition. Besides their theoretical importance, stochastic representations are extensively used in applications, e.g., to compute solutions through the particle tracking method (see Meerschaert et al. [2010]; Yong et al. [2006]; Meerschaert et al. [2010]). It is a deep and classical result that the solution to the diffusion equation

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x), & (t,x) \in (0,T] \times \mathbb{R}^d, \\ u(0,x) = \phi(0,x), & x \in \mathbb{R}^d, \end{cases}$$

allows the stochastic representation $u(t,x) = \mathbf{E}[\phi(0, B^x(t))]$. This normal diffusion model describes diffusion phenomena that exhibits homogeneity in both space and time. With the aid of single particle tracking, recent studies have provided many examples of anomalous diffusion.

[1]The results presented in this chapter are part of the joint work Du et al. [2018].

One typical example is the time-fractional (sub-)diffusion model,

$$\begin{cases} \partial_t^\beta u(t,x) = \Delta u(t,x), & (t,x) \in (0,T] \times \mathbb{R}^d, \\ u(0,x) = \phi(0,x), & x \in \mathbb{R}^d, \end{cases} \tag{3.3}$$

where ∂_t^β denotes the Caputo fractional derivative with order $\beta \in (0,1)$, which is defined by

$$\partial_t^\beta u(t) = \int_0^t \frac{(t-r)^{-\beta}}{\Gamma(1-\beta)} \partial_r u(r)\, dr.$$

The sub-diffusion phenomena has attracted much attention in applications such as contaminant transport in groundwater Kirchner et al. [2000], protein diffusion within cells Golding and Cox [2006], and thermal diffusion in fractal media Nigmatullin [1986]. The problem (3.3) has been extensively studied both analytically and numerically [Meerschaert and Sikorskii, 2012, Chapter 2.4]. Its solution can be expressed by $u(t,x) = \mathbf{E}[\phi(0, Y^x(t))]$ Meerschaert and Scheffler [2004], where $Y^x(t) = B^x(\tau_0^\beta(t))$ and $\tau_0^\beta(t) = \inf\{s > 0 : X^\beta(s) \geq t\}$ is the inverse process of the β-stable subordinator X^β. The density of $Y^x(t)$ can be derived using a conditioning argument Saichev and Zaslavsky [1997]; Baeumer and Meerschaert [2001]

$$H_{t,x}(y) = \int_0^\infty p_s(x,y) \partial_s \mathbf{P}[X^\beta(s) \geq t]\, ds, \tag{3.4}$$

where $\partial_s \mathbf{P}[X^\beta(s) \geq t] = \beta^{-1} t s^{-1-1/\beta} p_1^\beta(t s^{-1/\beta})$, with p_1^β being the density of $X^\beta(1)$ and $p_s(x)$ the density of $B^x(s)$. It is interesting to observe that the time-changed Brownian motion $Y^x(t)$ displays time heterogeneity, as the non-Markovian time change $t \mapsto \tau_0^\beta(t)$ is constant precisely when the subordinator $t \mapsto X^\beta(t)$ jumps Meerschaert and Sikorskii [2012]. This leads to the past-dependent diffusion Y^x being trapped, and in general spreading at a slower rate than B^x (see e.g. Zaslavsky [2002]; Piryatinska et al. [2005]; Magdziarz et al. [2007]). Moreover, the result can be generalised to other Caputo-type operators Meerschaert and Scheffler [2006]; Meerschaert et al. [2011]; Chen [2017]; Hernández-Hernández et al. [2017]. It is easy to see that the Caputo fractional derivative can be written in the form (3.2) by

$$\partial_t^\beta u(t) = c_\beta \int_0^\infty (u(t) - u(t-r)) r^{-\beta-1}\, dr,$$

with the kernel $\nu(t,r) := c_\beta r^{-\beta-1}$, where we extend the function u to the negative real line by $u(t) \equiv u(0)$ for $t \in (-\infty, 0)$. On the other hand, under certain hypothesis, one may show that the nonlocal operator could reproduce the first order derivative, as the horizon of nonlocal effects tends to zero Du et al. [2017]. Therefore, it is actually an interesting intermediate case between infinite-horizon fractional derivatives and infinitesimal local derivatives. Moreover, it can be shown that the nonlocal setting also serves to bridge between a short-time anomalous diffusion and a long-time normal diffusion Du and Zhou [2017], which has been observed in many experiments He et al. [2016].

Compared with the fractional diffusion model (3.3), the nonlocal-in-time model (3.1) requires a historical initial data, which could be time-dependent. As far as we know, the only work concerning the stochastic explanation of the historical initial data is Toniazzi [2019], which deals with the fractional case. In this chapter, we derive a stochastic representation of the solution to the problem (3.1) with a possibly time-dependent kernel ν and a historical initial data ϕ. As an example, we prove that the weak solution to the homogenous problem (for $f = 0$) allows the stochastic representation

$$\begin{aligned} u(t,x) &= \mathbf{E}\left[\phi\left(-X^{t,(\nu)}(\tau_0(t)), B^x(\tau_0(t))\right)\mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}}\right] \\ &= \int_{-\infty}^{0}\int_{\Omega}\phi(r,y)H_{t,x}(r,y)\,dr\,dy, \end{aligned} \tag{3.5}$$

where $\tau_0(t) = \inf\{s > 0 : -X^{t,(\nu)}(s) \le 0\}$, $\tau_\Omega(x) = \inf\{s > 0 : x + B(s) \notin \Omega\}$ and the heat kernel is given by

$$H_{t,x}(r,y) = \int_0^t \nu(z, z-r)\left(\int_0^\infty p_s^\Omega(x,y)\partial_z \mathbf{P}[-X^{t,(\nu)}(s) \le z]\,ds\right)dz.$$

Here we denote by $p_s^\Omega(x,y)$ the density of the killed Brownian motion $B^x(s)\mathbf{1}_{\{s<\tau_\Omega(x)\}}$. The representation (3.5) appears to be new, and it suggests an interesting interpretation. This is because the diffusion on Ω is still the subdiffusion $Y^x(t) = B^x(\tau_0(t))$, but the contribution in time of the initial condition $\phi(\cdot, Y^x(t))$ depends on the waiting/trapping time of $Y^x(t)$, which is indeed $W(t) = X^{t,(\nu)}(\tau_0(t))$.

The chapter is organized as follows. After reformulating the EE (3.1) into a Caputo-type fractional diffusion problem, we develop some general solution theory in Section 3.1, provided additional smoothness and compatibility conditions on problem data. In Section 3.2, we show that the stochastic representation provides a weak solution of (3.1) even though the data is weak. Finally, we present some numerical experiments to illustrate our theoretical findings.

Notation

In this chapter the spatial operator for the EE (3.1) is always the Dirichlet Laplacian, which we now define.

Definition 3.0.1. Let $\Omega \subset \mathbb{R}^d$ be a connected regular set. Let $(\Delta_\Omega, \mathrm{Dom}(\Delta_\Omega))$ be the generator of the Feller semigroup $P^\Omega = \{P_s^\Omega\}_{\ge 0}$ on $C_{\partial\Omega}(\Omega)$, where $P_s^\Omega f(x) := \mathbf{E}[f(B^x(s))\mathbf{1}_{\{s<\tau_\Omega(x)\}}]$, $s \ge 0$, $x \in \overline{\Omega}$, with $B^x(s) = x + B(2s)$, $s \ge 0$, $x \in \Omega$, $\{B(s)\}_{s\ge 0}$ being the standard d-dimensional Brownian motion. Also define the first exit times for $x \in \Omega$

$$\tau_\Omega(x) = \tau_\Omega^\Delta(x) = \inf\{s > 0 : B^x(s) \notin \Omega\}.$$

Remark 3.0.2. Recall that $\mathrm{Dom}(\Delta_\Omega) = \{f \in C_{\partial\Omega}(\Omega) \cap C^2(\Omega) : \Delta f \in C_{\partial\Omega}(\Omega)\}$ (see, e.g., [Baeumer et al., 2016b, Theorem 2.3]). We write $\Delta_\Omega = \Delta$ from now on. We denote the law of $B^x(s)\mathbf{1}_{\{s<\tau_\Omega(x)\}}$ by $p_s^\Omega(x,y)dy$, recalling that $(x,y) \mapsto p_s^\Omega(x,y)$ is continuous for each $s > 0$, and (H1b) holds.

Remark 3.0.3. For the arguments in Section 3.1 we could use in place of the Dirichlet Laplacian, the generator of Definition 1.3.14 with assumption (H1b).

For the time-derivatives we use the Feller triplets in Definition 1.3.10. We apply Theorem 1.6.3 for the spatial generator $(\Delta, \mathrm{Dom}(\Delta_\Omega))$ as in Definition 3.0.1. We maintain the notation of Theorem 1.6.3 the Feller triplets now obtained from Theorem 1.6.3-(i), Theorem 1.6.3-(ii) and Theorem 1.6.3-(iii).
Throughout this chapter, the notation c denotes a generic positive constant, whose value may differ at each occurrence.

3.1 Auxiliary generalised solution

In order to study the Feynman-Kac stochastic formula, we use following assumption on the initial data:

(H4) The initial data $\phi : (-\infty, 0] \times \overline{\Omega} \to \mathbb{R}$ is such that the extension of ϕ to $\phi(0)$ on $(0, T] \times \overline{\Omega}$ satisfies $\phi \in \mathrm{Dom}(\mathcal{L}^\infty_{(\nu),\Omega})$ and $\mathcal{L}^\infty_{(\nu),\Omega}\phi = (-D^{(\nu)}_\infty + \Delta)\phi$.

Remark 3.1.1. We have some observations on the assumption (H4):

(i) By Theorem 1.6.3-(i) and Definition 1.3.10-(i), assumption (H4) is satisfied for linear combinations of initial conditions in variable separable forms, that is, $\phi(t, x) = p(t)q(x)$, where $p \in C^1_\infty((-\infty, 0])$, $p'(0-) = 0$ and $q \in \mathrm{Dom}(\Delta_\Omega)$. Such set of functions is dense in $C_{\infty,\partial\Omega}((-\infty, 0] \times \Omega)$. The problem (3.1) with such a kind of initial data has been analytically studied in Du et al. [2017].

(ii) Note that (H4) implies $\phi(0) \in \mathrm{Dom}(\Delta_\Omega)$ and $f_\phi \in C([0, T] \times \Omega)$. This is because (H4) implies $\phi(0) \in C_{\partial\Omega}(\Omega)$, $\Delta\phi(t) = \Delta\phi(0) \in C_{\partial\Omega}(\Omega)$ for $t \in [0, T]$ and $f_\phi = -D^{(\nu)}_\infty\phi$.

(iii) The case where (H4) no longer holds is to be discussed in the next section.

We rewrite the Caputo-type EE for the comfort of the reader here

$$\begin{cases} D^{(\nu)}_0 u(t, x) = \Delta u(t, x) + g(t, x), & \text{in } (0, T] \times \Omega, \\ u(t, x) = \phi(0, x), & \text{in } \{0\} \times \Omega, \\ u(t, x) = 0, & \text{in } (0, T] \times \partial\Omega, \end{cases} \tag{3.6}$$

along with the definition of solutions as in Chapter 2.

Definition 3.1.2. Let $g \in C_{\partial\Omega}([0, T] \times \Omega)$ and $\phi(0) \in \mathrm{Dom}(\Delta_\Omega)$ such that $g(0) = -\Delta_\Omega\phi(0)$. We say that a function $u \in C_{\partial\Omega}([0, T] \times \Omega)$ is a *solution in the domain of the generator to problem* (3.6) if

$$\mathcal{L}_{(\nu),\Omega} u = -g \text{ on } (0, T] \times \overline{\Omega}, \quad u(0) = \phi(0), \quad \text{and } u \in \mathrm{Dom}(L_{(\nu),\Omega}). \tag{3.7}$$

Definition 3.1.3. Let $g \in B([0, T] \times \Omega)$ and $\phi(0) \in \mathrm{Dom}(\Delta_\Omega)$. We say that a function $u \in B([0, T] \times \Omega)$ is a *generalised solution to problem* (3.6) if

$$u = \lim_{n\to\infty} u_n \quad \text{pointwise,}$$

where $\{u_n\}_{n\in\mathbb{N}}$ is a sequence of of solutions in the domain of the generator for a corresponding sequence of data $\{g_n\}_{n\in\mathbb{N}} \subset C_{\partial\Omega}([0,T]\times\Omega)$ such that $g_n \to g$ a.e. on $(0,T]\times\Omega$, $\sup_n \|g_n\|_\infty < \infty$, and $g_n(0) = -\Delta_\Omega\phi(0)$ for each $n \in \mathbb{N}$.

Theorem 3.1.4. *Assume (H0). Then*

(i) If $g + \Delta\phi(0) \in C_{0,\partial\Omega}([0,T]\times\Omega)$ for some $g \in C_{\partial\Omega}([0,T]\times\Omega)$ and $\phi(0) \in Dom(\Delta_\Omega)$, then there exists a unique solution in the domain of the generator to problem (3.6).

(ii) Assume (H1a'). If $g \in B([0,T]\times\Omega)$ and $\phi(0) \in Dom(\Delta_\Omega)$, then there exists a unique generalised solution to problem (3.6), and the generalised solution allows the stochastic representation for any $(t,x) \in (0,T]\times\Omega$

$$\begin{aligned} u(t,x) = \mathbf{E}\left[\phi(0, B^x(\tau_0(t)))\mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}}\right] \\ + \mathbf{E}\left[\int_0^{\tau_0(t)\wedge\tau_\Omega(x)} g\left(-X^{t,(\nu)}(s), B^x(s)\right) ds\right]. \end{aligned} \tag{3.8}$$

(iii) Assume (H1a'), (H4) and let $g = f + f_\phi$, for $f \in B([0,T]\times\overline{\Omega})$. Then both solutions in part (i) and (ii) allow the stochastic representation for any $(t,x) \in (0,T]\times\Omega$

$$\begin{aligned} u(t,x) = \mathbf{E}\left[\phi\left(-X^{t,(\nu)}(\tau_0(t)), B^x(\tau_0(t))\right)\mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}}\right] \\ + \mathbf{E}\left[\int_0^{\tau_0(t)\wedge\tau_\Omega(x)} f\left(-X^{t,(\nu)}(s), B^x(s)\right) ds\right]. \end{aligned} \tag{3.9}$$

Proof. Parts (i) and (ii) follow immediately from Theorem 2.1.4-(i) and Theorem 2.1.4-(ii), respectively.

(iii) Extend ϕ to $\phi(0)$ on $(0,T]\times\overline{\Omega}$, and denote it again by ϕ. Then by Dynkin formula ([Dynkin, 1965, Theorem 5.1]) and Theorem 1.6.3-(iii) provided assumption (H4), we have

$$\mathbf{E}\left[\phi\left(-X^{t,(\nu)}(\tau_{t,x}), B^x(\tau_{t,x})\right)\right] - \phi(t,x) = \mathbf{E}\left[\int_0^{\tau_{t,x}} (-D_\infty^{(\nu)} + \Delta)\phi\left(-X^{t,(\nu)}(s), B^x(s)\right) ds\right].$$

Meanwhile, for $(t,x) \in (0,T]\times\overline{\Omega}$ the identities $f_\phi(t,x) = -D_\infty^{(\nu)}\phi(t,x)$, $\Delta\phi(0,x) = \Delta\phi(t,x)$ and

$$\int_0^t (\phi(t-r,x) - \phi(t,x))\nu(t,r)\,dr = \int_0^t (\phi(0,x) - \phi(0,x))\nu(t,r)\,dr = 0$$

hold, and we can derive the equality

$$\mathbf{E}\left[\int_0^{\tau_{t,x}} (-D_\infty^{(\nu)} + \Delta)\phi\left(-X^{t,(\nu)}(s), B^x(s)\right) ds\right] = \mathbf{E}\left[\int_0^{\tau_{t,x}} (f_\phi + \Delta\phi)\left(-X^{t,(\nu)}(s), B^x(s)\right) ds\right].$$

Therefore, the generalised solution allows the following representation

$$\begin{aligned}
u(t,x) &= \mathbf{E}\left[\int_0^{\tau_{t,x}} \Delta\phi\left(-X^{t,(\nu)}(s), B^x(s)\right) ds\right] + \phi(0,x) + \mathbf{E}\left[\int_0^{\tau_{t,x}} (f_\phi + f)\left(-X^{t,(\nu)}(s), B^x(s)\right) ds\right] \\
&= \mathbf{E}\left[\int_0^{\tau_{t,x}} (-D_\infty^{(\nu)} + \Delta)\phi\left(-X^{t,(\nu)}(s), B^x(s)\right) ds\right] + \phi(0,x) + \mathbf{E}\left[\int_0^{\tau_{t,x}} f\left(-X^{t,(\nu)}(s), B^x(s)\right) ds\right] \\
&= \mathbf{E}\left[\phi\left(-X^{t,(\nu)}(\tau_{t,x}), B^x(\tau_{t,x})\right)\right] + \mathbf{E}\left[\int_0^{\tau_{t,x}} f\left(-X^{t,(\nu)}(s), B^x(s)\right) ds\right] + \phi(0,x) - \phi(t,x) \\
&= \mathbf{E}\left[\phi\left(-X^{t,(\nu)}(\tau_0(t)), B^x(\tau_0(t))\right) \mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}}\right] + \mathbf{E}\left[\int_0^{\tau_{t,x}} f\left(-X^{t,(\nu)}(s), B^x(s)\right) ds\right].
\end{aligned}$$

for all $(t,x) \in (0,T] \times \Omega$. This completes the proof of the theorem. $\square$

Remark 3.1.5. Note that every generalised solution is the pointwise limit on $[0,T] \times \bar{\Omega}$ of a sequence of solutions in the domain of the generator $\{u_n\}_{n\in\mathbb{N}}$, and from the stochastic representation we can infer that $\sup_n \|u_n\|_{C([0,T]\times\bar{\Omega})} < \infty$. This implies the convergence $u_n \to u$ in $L^p((0,T)\times\Omega)$ for every $p \in [1,\infty)$.

We now give a more explicit formula for the heat kernel of the solution in (3.9) ($f = 0$).

Proposition 3.1.6. Let assumptions (H0) and (H1a') hold true. Then

$$\mathbf{E}\left[\phi\left(-X^{t,(\nu)}(\tau_0(t)), B^x(\tau_0(t))\right) \mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}}\right] = \int_{-\infty}^0 \int_\Omega \phi(r,y) H_{t,x}(r,y)\, dr\, dy, \tag{3.10}$$

for every $(t,x) \in (0,T] \times \Omega$ and $\phi \in B((-\infty,0] \times \Omega)$, where

$$H_{t,x}(r,y) = \int_0^t \nu(z, z-r) \left(\int_0^\infty p_s^\Omega(x,y) p_s^{(\nu)}(t,z)\, ds\right) dz.$$

Proof. By (H1a'), it is enough to prove formula (3.10) on the set $\{-X^{t,(\nu)}(\tau_0(t)) < 0\}$. Fix $(t,x) \in (0,T] \times \Omega$. Let $\phi \in \mathrm{Span}\{C_\infty^1(-\infty,T] \cdot \mathrm{Dom}(\Delta_\Omega)\}$ such that $\phi = 0$ on $[-n^{-1},T]$ for $n \in \mathbb{N}$. By Remark 3.1.1-(i) ϕ satisfies (H4). Then by Dynkin formula along with $\mathcal{L}_\Omega^{(\nu),\infty}\phi = (-D_\infty^{(\nu)} + \Delta)\phi$ by Theorem 1.6.3-(iii) and $\Delta\phi = 0$ on $(0,T]$, we have that

$$\begin{aligned}
u(t,x) :&= \mathbf{E}\left[\phi\left(-X^{t,(\nu)}(\tau_0(t)), B^x(\tau_0(t))\right) \mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}}\right] \\
&= \mathbf{E}\left[\int_0^{\tau_{t,x}} -D_\infty^{(\nu)}\phi\left(-X^{t,(\nu)}(s), B^x(s)\right)\, ds\right] \\
&= \int_0^\infty \mathbf{E}\left[\mathbf{1}_{\{s<\tau_0(t)\}} \int_{-X^{t,(\nu)}(s)}^\infty \phi\left(-X^{t,(\nu)}(s) - r, B^x(s\wedge\tau_\Omega(x))\right) \nu(-X^{t,(\nu)}(s), r)\, dr\right] ds
\end{aligned}$$

Next, using the independence of $-X^{t,(\nu)}(s\wedge\tau_0(t))$ and $B^x(s\wedge\tau_\Omega(x))$, $\{s < \tau_0(t)\} = \{0 <$

$-X^{t,(\nu)}(s)\}$, Fubini's Theorem and standard change of variables, we obtain

$$\begin{aligned} u(t,x) &= \int_\Omega \int_0^\infty \left(\int_0^t \left(\int_z^\infty \phi\,(z-r,y)\,\nu(z,r)\,dr \right) p_s^{(\nu)}(t,z)\,dz \right) p_s^\Omega(x,y)\,ds\,dy \\ &= \int_\Omega \int_0^\infty \left(\int_0^t \left(\int_{-\infty}^0 \phi\,(r,y)\,\nu(z,z-r)\,dr \right) p_s^{(\nu)}(t,z)\,dz \right) p_s^\Omega(x,y)\,ds\,dy \\ &= \int_{-\infty}^0 \int_\Omega \phi\,(r,y) \left(\int_0^t \nu(z,z-r) \int_0^\infty p_s^{(\nu)}(t,z) p_s^\Omega(x,y)\,ds\,dz \right) dy\,dr. \end{aligned}$$

By a density argument the identity (3.10) holds for every $\phi \in B((-\infty,n^{-1})\times\Omega)\cap C((-\infty,n^{-1})\times\Omega)$ for every $n\in\mathbb{N}$. Considering the non-negative increasing sequence $\phi_n = \mathbf{1}_{(-\infty,n^{-1})\times\Omega}$, $n\in\mathbb{N}$, by Monotone Convergence Theorem one can pass to the limit in both sides of (3.10), confirming that $H_{t,x}$ induces a finite measure on $(-\infty,0)\times\Omega$, as the right hand side of (3.10) is finite. By another density argument the equality (3.10) holds for every $\phi\in C_\infty((-\infty,0)\times\Omega)\cap\{f(0-)=f(x)=0,\ x\in\partial\Omega\}$, and we are done by Riesz-Markov-Kakutani representation Theorem [Kolokoltsov, 2011, Theorem 1.7.3]. $\square$

Remark 3.1.7. Suppose that (H0) and (H1a') hold, and that $\phi_n,\phi\in B((-\infty,0]\times\bar\Omega)$, for $n\in\mathbb{N}$, such that $\phi_n\to\phi$ a.e. on $(-\infty,0]\times\bar\Omega$, $\sup_n\|\phi_n\|_{B((-\infty,0]\times\bar\Omega)}<\infty$, and $f\in B((0,T]\times\Omega)$. Then Proposition 3.1.6 and Dominated Convergence Theorem imply that $u_n\to u$ pointwise on $(0,T]\times\Omega$ and $\sup_n\|u_n\|_{B((-\infty,0]\times\bar\Omega)}<\infty$. Here u_n is defined as (3.9) for ϕ_n, f, $n\in\mathbb{N}$, and u is defined as (3.9) for ϕ, f. This in turn implies the convergence $u_n\to u$ in $L^p((0,T)\times\Omega)$ for each $p\in[1,\infty)$.

3.2 Weak solution for the inhomogeneous Marchaud-type EE

In Section 3.1, the stochastic representation of the solution to the nonlocal-in-time evolution model (3.1) is established in case that the data is smooth and compatible. The aim of this section is to show that the representation (3.9) still provides a solution of (3.1) in the weak sense, even though the data does not satisfies the smoothness and compatibility conditions required in Section 3.1. In this section we use the slightly stronger assumption (H0') in place of (H0).

In case that the kernel is time-independent, i.e., $\nu(t,r)\equiv\nu(r)$, the existence and uniqueness of the weak solution (3.13) has been confirmed in Du et al. [2017]. The argument for the more general kernel in Theorem 3.2.10 is similar, so we only present some useful results here and omit some similar detailed proof in order to avoid redundancy.

Lemma 3.2.1. Suppose that $u\in B(-\infty,T)\cap L^1(-\infty,T)$, and $v\in C_c^\infty(0,T)$ with zero extension out of the interval $(0,T)$. Further, we suppose that

$$\int_0^T\int_0^\infty |u(t)-u(t-r)|\nu(t,r)\,dr\,dt<\infty. \tag{3.11}$$

Then it holds that

$$\int_0^T D_\infty^{(\nu)}u(t)v(t)\,dt = -\int_{-\infty}^T u(t)(D_\infty^{(\nu),*}v)(t)\,dt$$

with

$$D_\infty^{(\nu),*}v(t) = -\int_0^\infty v(t)\nu(t,r) - v(t+r)\nu(t+r,r)\,dr. \tag{3.12}$$

The next lemma gives an upper bound of $D_\infty^{(\rho)}$ for smooth functions in Sobolev spaces.

Lemma 3.2.2. Let the kernel ν satisfy (H0'). Then the operator $D_\infty^{(\rho)}$ defined by (3.2) satisfies

$$\|D_\infty^{(\rho)}v\|_{L^p(-\infty,T)} \le C\|v\|_{W^{1,p}(-\infty,T)}, \quad v \in W^{1,p}(-\infty,T).$$

with $p \in [1,\infty]$.

Proof. We only prove the result for $p \in [1,\infty)$, as the case $p = \infty$ follows analogously. By Hölder's inequality and assumption (H0') we have that for $p \in (1,\infty)$

$$\begin{aligned}
&\int_{-\infty}^T \left(\int_0^1 |u(t)-u(t-r)|\nu(t,r)\,ds\right)^p dt \\
&\le \int_{-\infty}^T \int_0^1 \frac{|u(t)-u(t-r)|^p}{r^p} r\nu(t,r)\,dr \left(\int_0^1 r\nu(t,r)\,dr\right)^{p-1} dt \\
&\le c\int_{-\infty}^T \int_0^1 \frac{|u(t)-u(t-r)|^p}{r^p} r\nu(t,r)\,ds\,dt \\
&\le c\int_0^1 r^{1-p}|\max_t \nu(t,r)| \int_{-\infty}^T |u(t)-u(t-r)|^p\,dt\,dr \\
&\le c\int_0^1 r|\max_t \nu(t,r)|\,dr\|u\|^p_{W^{1,p}(-\infty,T)} \le c\|u\|^p_{W^{1,p}(-\infty,T)},
\end{aligned}$$

where we apply the fact that $\int_{-\infty}^T |u(t)-u(t-r)|^p\,dt \le c|r|^p\|u\|^p_{W^{1,p}(-\infty,T)}$ in the second last inequality. On the other hand, we have the following estimate

$$\begin{aligned}
&\int_{-\infty}^T \left(\int_1^\infty |u(t)-u(t-r)|\nu(t,r)\,dr\right)^p dt \\
&\le \int_{-\infty}^T \int_1^\infty |u(t)-u(t-r)|^p\nu(t,r)\,dr \left(\int_1^\infty \nu(t,r)\,dr\right)^{p-1} dt \\
&\le c\int_{-\infty}^T \int_0^1 |u(t)-u(t-r)|^p\nu(t,r)\,dr\,dt \\
&\le c\int_1^\infty \max_t \nu(t,r) \int_{-\infty}^T |u(t)-u(t-r)|^p\,dt\,dr \\
&\le c\int_1^\infty \max_t \nu(t,r)\,dr\|u\|^p_{L^p(-\infty,T)} \le c\|u\|^p_{W^{1,p}(-\infty,T)}.
\end{aligned}$$

Then we obtain the desired assertion. □

Similar argument yields the following a priori bound for the dual operator $D_\infty^{(\nu),*}$ given by (3.12).

Lemma 3.2.3. Let the kernel ν satisfy (H0') and let the operator $D_\infty^{(\nu),*}$ be defined by (3.12).

Then for any $v \in W^{1,p}(\mathbb{R})$ with $p \in [1,\infty]$, it holds that

$$\|D_{\infty}^{(\nu),*} v\|_{L^p(\mathbb{R})} \le C\|v\|_{W^{1,p}(\mathbb{R})}.$$

Proof. First, we use the following splitting

$$D_{\infty}^{(\nu),*} v(t) = \int_0^{\infty} (v(t+r) - v(t))\nu(t,r)\, dr + \int_0^{\infty} v(t+r)(\nu(t,r) - \nu(t+r))\, dr = I_1 + I_2.$$

Now using the same argument as that in Lemma 3.2.2, we derive that for $p \in [1,\infty)$

$$\|I_1\|_{L^p(\mathbb{R})} \le C\|v\|_{W^{1,p}(\mathbb{R})}.$$

Therefore it suffices to bound I_2. For $p \in [1,\infty)$, by Hölder's inequality and assumption (H0') we have that

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left(\int_0^1 |v(t+r)||\nu(t,r) - \nu(t+r)|\, dr \right)^p dt \\
&\le \int_{-\infty}^{\infty} \int_0^1 |v(t+r)|^p |\nu(t,r) - \nu(t+r)|\, dr \left(\int_0^1 |\nu(t,r) - \nu(t+r,r)|\, dr \right)^{p-1} dt.
\end{aligned}$$

Then we observe that

$$\int_0^1 |\nu(t,r) - \nu(t+r,r)|\, dr \le \int_0^1 \int_t^{t+r} |\partial_y \nu(y,r)|\, dr \le \int_0^1 r \max_t |\partial_t \nu(t,r)|\, dr \le c,$$

and hence

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left(\int_0^1 |v(t+r)||\nu(t,r) - \nu(t+r,r)|\, dr \right)^p dt \\
&\le c \int_{-\infty}^{\infty} \int_0^1 |v(t+r)|^p |\nu(t,r) - \nu(t+r,r)|\, dr\, dt \\
&\le c \int_0^1 \int_{-\infty}^{\infty} |v(t+r)|^p\, dt \max_t |\nu(t,r) - \nu(t+r,r)|\, dr \\
&\le c\|v\|_{L^p(\mathbb{R})} \int_0^1 r \max_t |\partial_t \nu(t,r)|\, dr \le c\|v\|_{L^p(\mathbb{R})}.
\end{aligned}$$

Meanwhile, applying the following observation

$$\int_1^{\infty} |\nu(t,r) - \nu(t+r,r)|\, dr \le \int_1^{\infty} |\nu(t,r)| + |\nu(t+r,r)|\, dr \le c,$$

we have the following estimate

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left(\int_{1}^{\infty}|v(t+r)||\nu(t,r)-\nu(t+r,r)|\,ds\right)^{p}dt\\
&\le c\int_{-\infty}^{\infty}\int_{1}^{\infty}|v(t+r)|^{p}|\nu(t,r)-\nu(t+r,r)|\,ds\,dt\\
&\le c\int_{1}^{\infty}\int_{-\infty}^{\infty}|v(t+r)|^{p}\,dt(|\nu(t,r)|+|\nu(t+r,r)|)\,dr\\
&\le c\|v\|_{L^{p}(\mathbb{R})},
\end{aligned}$$

which yields that

$$\|I_2\|_{L^p(\mathbb{R})}\le C\|v\|_{W^{1,p}(\mathbb{R})}.$$

This completes the proof for $p\in[1,\infty)$, and the case that $p=\infty$ follows analogously. □

Then we have the following result for a smooth function with compact support.

Corollary 3.2.4. Let the kernel ν satisfy (H0') and let the operator $D_{\infty}^{(\nu),*}$ be defined by (3.12). Then $D_{\infty}^{(\nu),*}v\in L^1(-\infty,T)\cap L^{\infty}(-\infty,T)$ for any $v\in C_c^1(0,T)$.

We now introduce the notation $\langle u,v\rangle_a^b=\int_a^b u(t)v(t)\,dt$, $b>a\ge-\infty$ for the following definition.

Definition 3.2.5. We define the weak operator of a function $u\in L^1_{loc}(\mathbb{R})$ to be a function $\widetilde{D_{\infty}^{(\nu)}}u$ that satisfies

$$\langle\widetilde{D_{\infty}^{(\nu)}}u,v\rangle_{-\infty}^{T}=\langle u,D_{\infty}^{(\nu),*}v\rangle_{-\infty}^{T},\quad \text{for every } v\in C_c^{\infty}(0,T).$$

Lemma 3.2.6. Suppose that the kernel ν satisfies (H0') and it is variables-separable, i.e., $\nu(t,r)=p(t)q(r)$ with $p(t)\in C^1[0,T]$ and $p(t)\ge c_1>0$. Moreover, we let $u\in L^{\infty}(\mathbb{R})$ and $\widetilde{D_{\infty}^{(\nu)}}u\in L^2(0,T)$. Then $D_{\infty}^{(\rho)}u\in L^2(0,T)$ and

$$D_{\infty}^{(\rho)}u=\widetilde{D_{\infty}^{(\rho)}}u\quad\text{almost everywhere},$$

where $D_{\infty}^{(\rho)}$ is defined by (3.2).

Proof. First of all, we consider the case that the kernel function is translation preserved, i.e., $\nu(t,r)=\nu(r)$. To this end, we define the truncated nonlocal operator

$$D_{\delta}^{(\nu)}u(t)=\int_0^{\delta}(u(t)-u(t-r))\nu(r)\,dr$$

as well as its adjoint operator $D_{\delta}^{(\nu),*}$ and the weak operator $\widetilde{D_{\delta}^{(\nu)}}$. Since for any $\delta>0$, we have

$$\int_{\delta}^{\infty}(u(t)-u(t-r))\nu(r)\,dr=u(t)\int_{\delta}^{\infty}\nu(r)\,dr-\int_{\delta}^{\infty}u(t-r)\nu(r)\,dr\in L^2(0,T),$$

by assumption (H0'). By the definition of the weak operator, one may deduce that

$$\widetilde{D_{\delta}^{(\nu)}}u(t)=\widetilde{D_{\infty}^{(\nu)}}u(t)-\int_{\delta}^{\infty}(u(t)-u(t-r))\nu(r)\,dr\in L^2(0,T)$$

Now by Lemma [Du et al., 2017, Lemma 2.4] we have that $D_\delta^{(\nu)}u \in L^2(0,T)$ and $D_\delta^{(\nu)}u = \widetilde{D_\delta^{(\nu)}}u$. As a result, we derive that

$$D_\infty^{(\rho)}u(t) = \int_0^\infty (u(t) - u(t-r))\nu(r)\,dr = D_\delta^{(\nu)}u(t) + \int_\delta^\infty (u(t) - u(t-r))\nu(r)\,dr \in L^2(0,T),$$

and hence $D_\infty^{(\rho)}u = \widetilde{D_\infty^{(\rho)}}u$ almost everywhere.

Next, we consider the case that $\nu(t,r) = p(t)q(r)$ and define the operator

$$D_\infty^{(q)}u(t) = \int_0^\infty (u(t) - u(t-r))q(r)\,ds.$$

The same as before, we may define corresponding adjoint and weak operators. Then we note that

$$\langle p\widetilde{D_\infty^{(q)}}u, v\rangle_0^T = \langle u, D_\infty^{(q),*}(pv)\rangle_{-\infty}^T = \langle u, D_\infty^{(\nu),*}v\rangle_{-\infty}^T = \langle \widetilde{D_\infty^{(\nu)}}u, v\rangle_0^T,$$

which together with the positivity assumption on $p(t)$ yields that

$$\widetilde{D_\infty^{(q)}}u(t) = \frac{1}{p(t)}\widetilde{D_\infty^{(\nu)}}u(t) \le \frac{1}{c_1}\left|\widetilde{D_\infty^{(\nu)}}u(t)\right| \in L^2(0,T).$$

As a result, we obtain that $D_\infty^{(q)}u(t) = \widetilde{D_\infty^{(q)}}u(t) \in L^2(0,T)$ and

$$D_\infty^{(\rho)}u(t) = p(t)\widetilde{D_\infty^{(q)}}u(t) = \widetilde{D_\infty^{(\rho)}}u(t) \in L^2(0,T).$$

□

Lemma 3.2.7. Let $\tilde{u} \in B((-\infty,T] \times \Omega)$ be the function defined in (3.9) under the assumptions (H0') and (H1a'), for $\phi \in L^\infty(-\infty,0;H_0^1(\Omega))$ and $f \in L^\infty(0,T;H_0^1(\Omega))$. Then $\tilde{u} \in L^\infty(-\infty,T;H_0^1(\Omega))$.

Proof. Consider (3.9) for $f = 0$ (the proof for $f \neq 0$ is similar and omitted). Fix $t > 0$. By [Evans, 2010, Chapter 7.1] we have $T_s^\Omega\phi(r,\cdot) = \mathbf{E}[\phi(r,B^\cdot(s))\mathbf{1}_{\{s<\tau_\Omega\}}] \in H_0^1(\Omega)$ for a.e. $r \in (-\infty,0)$ and $s \ge 0$. Consider the Borel probability space (Γ,μ_t), where $\Gamma = (-\infty,0) \times (0,\infty)$ and $\mu_t(dsdr) = \left(\int_0^t \nu(z,z-r)p_s^{(\nu)}(t,z)\,dz\right)dsdr$, so that formula (3.10) reads $u(t,x) = \int_\Gamma T_s^\Omega\phi(r,x)\,\mu_t(dsdr)$. Note that for a.e. $r \in (-\infty,0)$ and every $s \ge 0$

$$\|T_s^\Omega\phi(r)\|_{H^1(\Omega)} \le \|\phi(r)\|_{H^1(\Omega)} \le \|\phi\|_{L^\infty(-\infty,0;H_0^1(\Omega))} =: C,$$

where the first inequality holds by [Evans, 2010, Chapter 7.1, Theorem 5.(i)], as $\phi(r) \in H_0^1(\Omega)$ for a.e. $r \in (-\infty,0)$. We conclude that $\tilde{u}(t) \in H_0^1(\Omega)$, because the above bound proves that $T_\cdot^\Omega\phi(\cdot) : (\Gamma,\mu_t) \to H_0^1(\Omega)$ is Bochner integrable, which implies that $\tilde{u}(t) = \int_\Gamma T_\cdot^\Omega\phi(\cdot)\,\mu_t(d\cdot) = \lim_{n\to\infty} S_n$ in $H^1(\Omega)$, where each S_n is a linear combination of functions in $H_0^1(\Omega)$.

Formula (3.10) suggests the definition

$$\begin{aligned}\nabla\tilde{u}(t,x) :&= \int_{-\infty}^{0}\left(\int_{0}^{t}\nu(z,z-r)\left(\int_{0}^{\infty}\nabla T_s^{\Omega}\phi(r,x)p_s^{(\nu)}(t,z)\,ds\right)dz\right)dr\\ &= \int_{\Gamma}\nabla T_s^{\Omega}\phi(r,x)\,\mu_t(dsdr).\end{aligned}$$

Then $\nabla\tilde{u}(t)\in L^2(\Omega)$, because

$$\begin{aligned}\int_{\Omega}(\nabla\tilde{u}(t,x))^2\,dx &= \int_{\Omega}\left(\int_{\Gamma}\nabla T_s^{\Omega}\phi(r,x)\,\mu_t(dsdr)\right)\left(\int_{\Gamma}\nabla T_{s'}^{\Omega}\phi(r',x)\,\mu_t(ds'dr')\right)dx\\ &= \int_{\Gamma}\int_{\Gamma}\left(\int_{\Omega}\nabla T_s^{\Omega}\phi(r,x)\nabla T_{s'}^{\Omega}\phi(r',x)\,dx\right)\mu_t(dsdr)\mu_t(ds'dr')\\ &\le \int_{\Gamma}\int_{\Gamma}\|T_s^{\Omega}\phi(r)\|_{H^1(\Omega)}\|T_{s'}^{\Omega}\phi(r')\|_{H^1(\Omega)}\,\mu_t(dsdr)\mu_t(ds'dr')\\ &\le C^2\left(\int_{\Gamma}\mu_t(dsdr)\right)^2 = C^2.\end{aligned}$$

Applying Fubini's Theorem to the definition of weak derivative proves that $\nabla\tilde{u}(t)$ is indeed the weak derivative of $\tilde{u}(t)$. Finally, $\sup_{t\in(0,T)}\int_{\Omega}(\nabla\tilde{u}(t,x))^2\,dx\le C^2$ and the smoothness of ϕ implies that $\tilde{u}\in L^{\infty}(-\infty,T;H_0^1(\Omega))$, concluding the proof. □

Next we shall show that the stochastic representation (3.9) provides the weak solution of problem (3.1), whose definition is given as below.

Definition 3.2.8. A function u is called a *weak solution to problem* (3.1) if $u\in L^2(0,T;H_0^1(\Omega))$ and $\widetilde{D_{\infty}^{(\nu)}}u\in L^2(0,T;H^{-1}(\Omega))$, and for every $v\in L^2(0,T;H_0^1(\Omega))$ (with zero extension to $t<0$)

$$\begin{cases}\langle\widetilde{D_{\infty}^{(\nu)}}u,v\rangle = -\langle\nabla u,\nabla v\rangle+\langle f,v\rangle, & \text{and,}\\ \qquad u(t)=\phi(t), & \text{for a.e. } t\in(-\infty,0),\end{cases}\tag{3.13}$$

where the notation $\langle\cdot,\cdot\rangle$ is defined by

$$\langle u,v\rangle = \int_{-\infty}^{T}\int_{\Omega}u(t,x)v(t,x)\,dx\,dt,$$

or the duality

$$\langle u,v\rangle = \int_{-\infty}^{T}(u(t),v(t))\,dt,$$

in case that $u\in L^2(0,T;H^{-1}(\Omega))$, where $(\cdot,\cdot)$ is the dual pairing of $H_0^1(\Omega)$.

Remark 3.2.9. If u is the weak solution of (3.1) and $\widetilde{D_{\infty}^{(\nu)}}u\in L^2(0,T;L^2\Omega)$, we have $\widetilde{D_{\infty}^{(\nu)}}u = D_{\infty}^{(\nu)}u$ by Lemma 3.2.6, provided that the kernel function is variables-separable, i.e., $\nu(t,s)=p(t)q(s)$ with $p(t)\in C^1[0,T]$ and $p(t)\ge c_1>0$. Then u satisfies the equation (4.1) almost everywhere.

Theorem 3.2.10. *Assume (H0') and (H1a'). Let u be given by formula (3.9), where $\phi\in L^{\infty}(-\infty,0;H_0^1(\Omega))\cap L^{\infty}((-\infty,0)\times\Omega)$ and $f\in L^{\infty}(0,T;H_0^1(\Omega))\cap L^{\infty}((0,T)\times\Omega)$. Define the* *extension* $\tilde{u}$ *of* u *as*

$$\tilde{u} := \begin{cases} u, & \text{on } (0,T] \times \Omega, \\ \phi, & \text{on } (-\infty, 0) \times \Omega. \end{cases} \tag{3.14}$$

Then $\tilde{u}$ *is a weak solution to problem (3.1).*

Proof. Assume for the first two steps that ϕ satisfies (H4).

Step 1: Let u be a solution in the domain of the generator to problem (3.6) for $g \equiv f + f_\phi$, and initial condition $\phi(0)$, for some $f \in C_{\partial\Omega}([0,T] \times \Omega)$. As $u \in \mathrm{Dom}(\mathcal{L}_{(\nu),\Omega})$, by Theorem 1.6.3-(iv), $u - \phi(0) \in \mathrm{Dom}(\mathcal{L}^{\mathrm{kill}}_{(\nu),\Omega})$, and hence applying Theorem 1.6.3-(ii) there exists $\{\hat{u}_n\}_{n\in\mathbb{N}} \subset \mathrm{Dom}(\mathcal{L}^{\mathrm{kill}}_{(\nu),\Omega})$ such that

$$\hat{u}_n \to u - \phi(0), \quad \mathcal{L}_{(\nu),\Omega}\hat{u}_n \to \mathcal{L}_{(\nu),\Omega}(u - \phi(0)) \ \text{ and } \ \mathcal{L}_{(\nu),\Omega}\hat{u}_n = (-D^{(\nu)}_0 + \Delta)\hat{u}_n.$$

Then we apply Theorem 1.6.3-(i)-(iv) to obtain that

$$u_n := \hat{u}_n + \phi(0) \in \mathrm{Dom}(\mathcal{L}_{(\nu),\Omega}), \quad u_n \to u, \quad \mathcal{L}_{(\nu),\Omega}u_n = \mathcal{L}_{(\nu),\Omega}\hat{u}_n + \Delta\phi(0) \to \mathcal{L}_{(\nu),\Omega}u$$

and $u_n(0) = \phi(0)$ for all $n \in \mathbb{N}$. Then using the fact that $D^{(\nu)}_\infty \tilde{u}_n = D^{(\nu)}_0 u_n - f_\phi$ for $t \in [0,T]$ we have

$$(D^{(\nu)}_0 - \Delta)u_n - f_\phi = D^{(\rho)}_\infty \tilde{u}_n - \Delta\tilde{u}_n, \quad \text{on } [0,T] \times \Omega,$$

where $\tilde{u}_n$ is defined for each $n \in \mathbb{N}$ by

$$\tilde{u}_n := \begin{cases} u_n, & \text{on } (0,T] \times \Omega, \\ \phi, & \text{on } (-\infty, 0] \times \Omega. \end{cases} \tag{3.15}$$

Therefore, we have that

$$(-D^{(\nu)}_\infty + \Delta)\tilde{u}_n = (-D^{(\nu)}_0 + \Delta)u_n + f_\phi \to \mathcal{L}_{(\nu),\Omega}u + f_\phi = -f,$$

where the convergence is in $C_{\partial\Omega}([0,T] \times \Omega)$.

On the other hand, we apply Corollary 3.2.4 for any $v \in C^\infty_c((0,T) \times \Omega)$ to obtain as $n \to \infty$

$$\langle(-D^{(\nu)}_\infty + \Delta)\tilde{u}_n, v\rangle = \langle\tilde{u}_n, (-D^{(\nu),*}_\infty + \Delta)v\rangle \to \langle\tilde{u}, (-D^{(\nu),*}_\infty + \Delta)v\rangle,$$

where Corollary 3.2.4 guarantees that $(-D^{(\nu),*}_\infty + \Delta)v \in L^1((-\infty,0) \times \Omega) \cap L^\infty((0,T) \times \Omega)$, and hence

$$\langle u, (D^{(\nu),*}_\infty - \Delta)v\rangle = \langle f, v\rangle, \qquad \text{for any } \ v \in C^\infty_c((0,T) \times \Omega).$$

Step 2: Let now u be the generalised solution to problem (3.6) for $g = f + f_\phi$, where $f \in L^\infty((0,T) \times \Omega) \cap L^\infty(0,T; H^1_0(\Omega))$, and let $\tilde{u}$ be its extension with historical initial data ϕ. By the definition of the generalised solution, we pick a sequence $f_n \in C_{\partial\Omega}([0,T] \times \Omega)$ such that

$$f_n \to f \quad \text{a.e.,} \qquad f_n(0) = -(f_\phi(0) + \Delta\phi(0)) \qquad \text{and} \qquad \sup_n \|f_n\|_\infty < \infty.$$

Besides, we denote by u_n the respective solution in the domain of the generator and let $\tilde{u}_n$ be its extension by (3.15). Then by Step 1, we know that each $\tilde{u}_n$ satisfies

$$\langle \tilde{u}_n, (-D_{\infty}^{(\nu),*} + \Delta)v\rangle = \langle -f_n, v\rangle, \qquad \text{for any} \quad v \in C_c^{\infty}((0,T)\times\Omega),$$

as well as the initial and boundary conditions in (3.1). Now the Dominated Convergence Theorem provided the uniform upper bound of f_n implies that

$$f_n \to f \qquad \text{in} \qquad L^2(0,T;L^2\Omega) \qquad \text{as} \quad n\to\infty.$$

On the other hand, we have $\tilde{u}_n \to \tilde{u}$ in $L^2(0,T;L^2\Omega)$ by Remark 3.1.5. Meanwhile $(D_{\infty}^{(\nu),*} - \Delta)v \in L^1((-\infty,0)\times\Omega)\cap L^{\infty}((0,T)\times\Omega)$ for any $v\in C_c^{\infty}((0,T)\times\Omega)$ by Corollary 3.2.4. Therefore we obtain as $n\to\infty$

$$\langle \tilde{u}_n, (D_{\infty}^{(\nu),*} - \Delta)v\rangle \to \langle \tilde{u}, (D_{\infty}^{(\nu),*} - \Delta)v\rangle, \qquad \text{for any} \quad v \in C_c^{\infty}((0,T)\times\Omega).$$

Step 3: Now we consider the case that $\phi \in L^{\infty}((-\infty,0)\times\Omega))\cap L^{\infty}(-\infty,0;H_0^1(\Omega))$ and $f\in L^{\infty}(0,T;H_0^1(\Omega))\cap L^{\infty}((0,T)\times\Omega)$. To this end, we set functions $\phi_K(t,x) = \phi(t,x)\mathbf{1}_{\{t<-K\}}$, for $K\in\mathbb{N}$. By the density of $\mathrm{Span}\{C_c^{\infty}((-K,0))\cdot C_c^{\infty}(\Omega)\}$ in $B([-K,0]\times\overline{\Omega})$ with respect to sequential convergence a.e., we choose $\phi_{K,j}\in \mathrm{Span}\{C_c^{\infty}((-K,0))\cdot C_c^{\infty}(\Omega)\}$ such that

$$\phi_{K,j}\to\phi_K \quad \text{a.e.} \quad \text{and} \quad \sup_j \|\phi_{K,j}\|_{\infty} < \infty.$$

By Remark 3.1.1-(i), we know that $\phi_{K,j}$ satisfies assumption (H4) for each $j\in\mathbb{N}$. Denote by $u_{K,j}$ the generalised solution with the initial data $\phi_{K,j}$ and source term f, and denote by u_K the function given by formula (3.9) with $\phi\equiv\phi_K$ and source term f. By Remark 3.1.7 we conclude that

$$\sup_j \|\tilde{u}_{K,j}\|_{\infty} < \infty \quad \text{and} \quad \tilde{u}_{K,j}\to\tilde{u}_K \quad \text{a.e. on} \quad (-K,T]\times\Omega.$$

Then for any $v\in C_c^{\infty}((0,T)\times\Omega)$, we know that $(D_{\infty}^{(\nu)} - \Delta)^* v \in L^1((-K,0)\times\Omega)\cap L^{\infty}((0,T)\times\Omega)$ by Corollary 3.2.4, and hence

$$\langle \tilde{u}_K, (D_{\infty}^{(\nu),*} - \Delta)v\rangle = \lim_{j\to\infty}\langle \tilde{u}_{K,j}, (D_{\infty}^{(\nu),*} - \Delta)v\rangle = \langle f, v\rangle, \tag{3.16}$$

and $\tilde{u}_K = \phi_K$ on $(-K,0]\times\Omega$. We can now pass to the limit as $K\to\infty$ in (3.16), given that $\tilde{u}_K \to \tilde{u}$ a.e. on $(-\infty,T)\times\Omega$, with $\sup_K\|\tilde{u}_K\|_{\infty} < \infty$, again by Remark 3.1.7, and $(D_{\infty}^{(\nu)} - \Delta)^* v\in L^1((-\infty,0)\times\Omega)\cap L^{\infty}((0,T)\times\Omega)$ by Corollary 3.2.4. Here u is defined by (3.9) for ϕ and f, and $\tilde{u}$ by (4.3.5). Therefore we conclude that

$$\langle \tilde{u}, (D_{\infty}^{(\nu),*} - \Delta)v\rangle = \langle f, v\rangle.$$

And so

$$\langle \widetilde{D_{\infty}^{(\nu)}}\tilde{u}, v\rangle + \langle\nabla\tilde{u}, \nabla v\rangle = \langle f, v\rangle,$$

by Lemma 3.2.7 and the smoothness of the problem data f and ϕ, confirming that $\tilde{u}$ is a weak solution to problem (3.1). $\square$

Remark 3.2.11. If $\phi(t,x) \equiv \phi_0(x) \in H_0^1(\Omega)$ in Theorem 3.2.10, then one recovers the weak solution to the (inhomogeneous) Caputo-type fractional diffusion equation Chen [2017]; Hernández-Hernández et al. [2017]

$$u(t,x) = \mathbf{E}\left[\phi_0\left(B^x(\tau_0(t)\right)\mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}}\right] + \mathbf{E}\left[\int_0^{\tau_0(t)\wedge\tau_\Omega(x)} f\left(-X^{t,(\nu)}(s), B^x(s)\right) ds\right].$$

Remark 3.2.12. The solution in Theorem 3.2.10 will be continuous at $t = 0$ for every $x \in \Omega$ if ϕ is continuous at every point in $\{0\} \times \Omega$ and $\tau_0 : [0,T] \to \mathbb{R}$ is continuous. This can be proved by a stochastic continuity argument for the first term of the solution (3.9), and for the second term one can use $\mathbf{E}[\tau_0(t)] \to 0$ as $t \downarrow 0$ (which is a consequence of the continuity of τ_0). However, the solution (3.9) will in general fail to be continuous at $t = 0$ even for smooth data. This is for example the case of integrable kernels $\int_0^\infty \nu(r)\, dr < \infty$ (see [Toniazzi, 2019, Remark A.3]).

Chapter 4

Marchaud EE: stochastic classical solution

It is a classical result that the solution to the standard heat equation $\partial_t u = \Delta u$, $u(0) = \phi_0$ allows the stochastic representation $u(t,x) = \mathbf{E}[\phi_0(X^{x,2}(t))]$, where $X^{x,2}$ is a Brownian motion started at $x \in \mathbb{R}^d$. Space-time fractional evolution equations (EEs) extend the heat equation by introducing space-time heterogeneity. This often is done by considering the Caputo EE $D_0^\beta u = -(-\Delta)^{\frac{\alpha}{2}} u$, where one substitutes the local operators ∂_t and Δ with fractional analogues. Respectively, the Caputo derivative $D_0^\beta u(t) = c_\beta \int_0^t u'(r)(t-r)^{-\beta}\, dr$ and the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}} u(x) = \mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}u(\xi))(x)$, where $\beta \in (0,1)$, $\alpha \in (0,2)$, $c_\beta = \Gamma(1-\beta)^{-1}$ and $\mathcal{F}$ is the Fourier transform (for standard references see Diethelm [2010]; Bogdan et al. [2009]). It is well known that the fundamental solution to the Caputo EE is the law of the non-Markovian anomalous diffusion $Y^x(t) = X^{x,\alpha}(\tau_0(t))$ (see, e.g., Meerschaert and Sikorskii [2012]). Here $X^{x,\alpha}$ is the rotationally symmetric α-stable Lévy process started at $x \in \mathbb{R}^d$ and $\tau_0(t)$ is the inverse process of the β-stable subordinator $X^\beta(t)$. The density of this beautiful formula was first observed in Saichev and Zaslavsky [1997]. The time change interpretation first appeared in Meerschaert et al. [2002]; Meerschaert and Scheffler [2004], based on Baeumer and Meerschaert [2001]. The process Y^x displays space-heterogeneity due to the jump nature of $X^{x,\alpha}$. Also time-heterogeneity features in Y^x, as the time change $t \mapsto \tau_0(t)$ is constant precisely when the subordinator $t \mapsto X^\beta(t)$ jumps, so that $t \mapsto Y^x(t)$ is trapped on such time intervals. This interesting trapping phenomenon leads to the process Y^x spreading at a slower rate than $X^{x,\alpha}$. Indeed, in the physics literature the anomalous diffusion Y^x is often referred to as a sub-diffusion when $\alpha = 2$ (see, e.g., Zaslavsky [1994]; Piryatinska et al. [2005]; Magdziarz et al. [2007]). See Meerschaert and Scheffler [2004] for a characterisation of Y^x as the scaling limit of continuous time random walks with heavy-tailed waiting times. See Barlow and Černý [2011] for a characterisation of Y^x as the scaling limit of random conductance models or asymmetric Bouchaud's trap models ($\alpha = 2$). See Magdziarz [2010]; Magdziarz and Schilling [2015] for sample path properties of Y^x, and Deng and Schilling [2018]; Chen et al. [2018] for heat kernel asymptotic formulas. Existence of classical solutions for Caputo EEs is generally a subtle problem. The works Eidelman and Kochubei [2004]; Baeumer et al. [2009]; Allen et al. [2016] tackle classical solutions on unbounded domains. Meanwhile the works Chen et al. [2012]; Meerschaert et al. [2009, 2011]; Leonenko et al. [2013]

consider bounded domains, and all their proofs rely on the spectral decomposition of the spatial operator. Stochastic representations for solutions to time-nonlocal equations is an active area of theoretical research (see, e.g., Baeumer et al. [2016b]; Chen [2017]; Hernández-Hernández et al. [2017]; Chen et al. [2018]). Partly because they provide formulas in the general absence of closed forms along with suggesting probabilistic proof methods. Moreover, such representations can be useful for particle tracking codes (see, e.g., Meerschaert et al. [2010]). Let us remark that Caputo EEs are applied in a variety of fields, such as physics, finance, economics, biology and hydrogeology (see, e.g., Zaslavsky [2002]; Scalas [2006]; Scalas et al. [2000]; Benson et al. [2013]; Fedotov and Iomin [2008]).

In this chapter we focus on the following extension of the Caputo EE: the inhomogeneous space-time fractional EE on bounded domain with Dirichlet boundary conditions and time-nonlocal initial condition

$$\begin{cases} D_\infty^\beta \tilde{u}(t,x) = \Delta_\Omega^{\frac{\alpha}{2}} \tilde{u}(t,x) + g(t,x), & \text{in } (0,T]\times\Omega, \\ \quad \tilde{u}(t,x) = 0, & \text{in } [0,T]\times\partial\Omega, \\ \quad \tilde{u}(t,x) = \phi(t,x), & \text{in } (-\infty,0]\times\Omega, \end{cases} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^d$ is a regular domain, $\Delta_\Omega^{\frac{\alpha}{2}}$ is the restricted fractional Laplacian[1], and the time operator $-D_\infty^\beta$ is the generator of the inverted β-stable subordinator[2]

$$D_\infty^\beta f(t) = \int_0^\infty (f(t-r) - f(t)) \frac{\Gamma(-\beta)^{-1} dr}{r^{1+\beta}}, \quad t \in \mathbb{R}. \tag{4.2}$$

As the main result of this work we prove existence and uniqueness of classical solutions to problem (4.1) along with the stochastic representation for the solution

$$\begin{aligned} \tilde{u}(t,x) = \mathbf{E}\left[\phi\left(-X^{t,\beta}(\tau_0(t)), X^{x,\alpha}(\tau_0(t))\right)\mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}}\right] \\ + \mathbf{E}\left[\int_0^{\tau_0(t)\wedge\tau_\Omega(x)} g\left(-X^{t,\beta}(s), X^{x,\alpha}(s)\right) ds\right], \end{aligned} \tag{4.3}$$

where the processes $-X^{t,\beta} = t - X^\beta$ and $X^{x,\alpha}$ are independent, and $\tau_\Omega(x)$ is the first exit time of $X^{x,\alpha}$ from Ω. To see why problem (4.1) extends the Caputo EE, let $\phi(t) = \phi(0)$ for every $t \in (-\infty, 0)$ and $g = 0$ in both (4.1) and (4.3). Then

$$D_\infty^\beta \tilde{u}(t) = \int_0^t (\tilde{u}(t-r) - \tilde{u}(t)) \frac{\Gamma(-\beta)^{-1} dr}{r^{1+\beta}} - \frac{\phi(0) - \tilde{u}(t)}{\Gamma(1-\beta)} t^{-\beta} = D_0^\beta u(t),$$

where u is the restriction of $\tilde{u}$ to $t \geq 0$, and one obtains the homogeneous Caputo EE and its solution, respectively. The recent works Chen et al. [2017]; Du et al. [2017] introduced a class

[1] We define $\Delta_\Omega^{\frac{\alpha}{2}}$ on functions on Ω, so that the Euclidean boundary $\partial\Omega$ makes sense in (4.1). In the literature the operator $\Delta_\Omega^{\frac{\alpha}{2}}$ is often defined through the application of the singular integral definition of $-(-\Delta)^{\frac{\alpha}{2}}$ to functions vanishing outside Ω (see, e.g., Bonforte and Vázquez [2016]).

[2] The operator D_∞^β is often referred to as the Marchaud derivative in the fractional calculus literature (see, e.g., Samko and Marichev [1993]).

of EEs that formally includes (4.1). They are motivated by the success of related nonlocal EEs arising in image processing, peridynamics and heat conduction (see, e.g., Gilboa and Osher [2008]; Bobaru and Duangpanya [2010]; Silling and Lehoucq [2010]; Du et al. [2012]), and the general lack of alternatives to Caputo-type time-nonlocal models. Part of their intent is to introduce initial conditions on the 'past' (ϕ on $(-\infty, 0) \times \Omega$). Our stochastic solution (4.3) appears to be new, and it provides an interesting interpretation for the time-nonlocal initial condition ϕ. This is because the overshoot $W(t) = X^{t,\beta}(\tau_0(t))$ is the waiting/trapping time of the anomalous diffusion $X^{x,\alpha}(\tau_0(t))$. We discuss an interpretation where the values of ϕ on $(-\infty, 0) \times \Omega$ describe the initial condition at time 0 with respect to the 'depth' of Ω, rather than the 'past' of Ω. To the best of our knowledge, there are no classical-wellposedness results for the EE (4.1). Related weak-wellposedness results can be found in Chen et al. [2017]; Du et al. [2017] (for certain general Lévy kernels in (4.2)) and indirectly in Liao [1989] (for abstract Markovian generators), meanwhile Allen [2017] considers uniqueness of weak solutions. Worth mentioning that our simple Lemma 4.3.5 allows to obtain wellposedness and regularity results for EEs such as (4.1) as corollaries of theorems concerning inhomogeneous Caputo EEs (see, e.g., Eidelman and Kochubei [2004]; Allen et al. [2016]). To see why the stochastic representation (4.3) is natural, one can formally apply the classical probabilistic intuition for elliptic boundary value problems (see, e.g., [Dynkin, 1965, Introduction, §3]) to problem (4.1) rewritten as

$$\begin{cases} \mathcal{L}\tilde{u} = -g, & \text{in } \Gamma, \\ \tilde{u} = \phi, & \text{in } \partial\Gamma, \end{cases} \tag{4.4}$$

where $\mathcal{L} = (-D^{\beta}_{\infty} + \Delta^{\frac{\alpha}{2}}_{\Omega})$ is the generator of the process $\{(-X^{t,\beta}(s), X^{x,\alpha}(s))\mathbf{1}_{\{s<\tau_\Omega(x)\}}\}_{s\geq 0}$ taking values in $(-\infty, T] \times \Omega$, $\Gamma = (0, T] \times \Omega$, and $\partial\Gamma := (-\infty, 0] \times \Omega \cup [0, T] \times \partial\Omega$, with $\phi = 0$ on $(0, T] \times \partial\Omega$.

To prove our main result, Theorem 4.3.6, we derive two results of independent interest. Namely:

- Theorem 4.2.6: the stochastic representation

$$\begin{aligned} u(t,x) = \mathbf{E}\left[\phi_0\left(X^{x,\alpha}(\tau_0(t))\right)\mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}}\right] \\ + \mathbf{E}\left[\int_0^{\tau_0(t)\wedge\tau_\Omega(x)} f\left(-X^{t,\beta}(s), X^{x,\alpha}(s)\right) ds\right], \end{aligned} \tag{4.5}$$

  is the unique classical solution to the inhomogeneous Caputo EE on bounded domain

$$\begin{cases} D^{\beta}_{0} u(t,x) = \Delta^{\frac{\alpha}{2}}_{\Omega} u(t,x) + f(t,x), & \text{in } (0,T] \times \Omega, \\ u(t,x) = 0, & \text{in } [0,T] \times \partial\Omega, \\ u(t,x) = \phi_0(x), & \text{in } \{0\} \times \Omega; \end{cases} \tag{4.6}$$

- Theorem 4.1.9: the stochastic representation (4.5) is a weak solution to problem (4.6).

Let us outline our proof strategy for Theorem 4.3.6. By plugging the values of ϕ in $\tilde{u}$, it is not hard to show the equivalence of classical solutions to problem (4.1) and to problem (4.6) with forcing term $f = g - D^\beta_\infty \phi$ and initial condition $\phi_0 = \phi(0)$ (see Lemma 4.3.5). Moreover, a Dynkin formula argument proves that the respective stochastic representations (4.3) and (4.5) agree (see Lemma 4.3.1). Hence, it is enough to prove Theorem 4.2.6. We do so by proving Theorem 4.1.9 and then showing the required regularity of the candidate solution (4.5). The main feature of our regularity assumption on the data ϕ and g is the differentiability in time. This is a consequence of the regularity assumption on f in Theorem 4.2.6, which we discuss now. Theorem 4.2.6 extends the proof of [Chen et al., 2012, Theorem 5.1], where problem (4.6) is treated for $f = 0$. This proof uses separation of variables combing eigenfunction expansions of $\Delta_\Omega^{\frac{\alpha}{2}}$ with Mittag-Leffler solutions to the Caputo initial value problem. Our separation of variables formula for the second term in (4.5) reads

$$\sum_{n=1}^{\infty} \psi_n(x) u_n(t) = \sum_{n=1}^{\infty} \psi_n(x) \int_0^t \langle f(s), \psi_n \rangle (t-s)^{\beta-1} \beta E'_\beta(-\lambda_n (t-s)^\beta)\, ds,$$

where $E_\beta(t) = \sum_{k=0}^{\infty} t^k \Gamma(k\beta+1)^{-1}$ is a Mittag-Leffler function, $\{\lambda_n, \psi_n\}_{n\in\mathbb{N}}$ is the system of eigenvalues-eigenfunctions of $\Delta_\Omega^{\frac{\alpha}{2}}$ and $\langle \cdot, \cdot \rangle$ is the inner product on Ω. Unsurprisingly, each u_n is the solution to the inhomogeneous Caputo initial value problem $D_0^\beta u_n(t) = -\lambda_n u_n(t) + \langle f(t), \psi_n \rangle$, $u_n(0) = 0$ (see [Diethelm, 2010, Theorem 7.2]). As we require differentiability of $t \mapsto u(t)$, we want to differentiate each $t \mapsto u_n(t)$. To compensate for the singularity of the Mittag-Leffler kernel $t^{\beta-1} E'_\beta(-\lambda_n t^\beta)$ we require differentiability of $t \mapsto f(t)$. Note that for the space fractional heat equation ($\beta = 1$) the Mittag-Leffler kernel is an exponential, and so continuity of f is enough to differentiate the u_n's. Related results in the literature also require differentiability on f (see, e.g, [Allen et al., 2016, Theorem 7.3]). Briefly, the arguments for Theorem 4.1.9 reduce the Caputo EE (4.6) to a Poisson equation with zero boundary conditions on $\{0\} \times \Omega \cup [0,T] \times \partial\Omega$ by constructing space-time Feller semigroups. We rely on the fact that the generator $-D_0^\beta$ only requires boundary conditions on the trivial set $\{0\}$. These arguments are an extension of the ideas in Hernández-Hernández et al. [2017], and they appear versatile. For example, they can be used to prove stochastic weak solutions for problem (4.1) with general nonlocal operators in both space and time (ongoing work with the authors in Du et al. [2017]). As far as we know, stochastic representations for solutions such as (4.5) for time-nonolocal EEs appear in Hernández-Hernández et al. [2017], meanwhile in Baeumer et al. [2005] the solution is given a representation via the superposition principle. Possibly worth mentioning that we do not invoke [Baeumer and Meerschaert, 2001, Theorem 3.1] and all our methods work for the standard Laplacian case $\alpha = 2$.

This chapter is structured as follows: in Section 4.1 we prove Theorem 4.1.9. In Section 4.2 we prove Theorem 4.2.6. In Section 4.3 we prove that the stochastic representation (4.3) is the unique classical solution to the EE (4.1).

Notation

In this chapter we use the notation in Section 1.4, recalling the adjustments in Remark 1.4.10. By Proposition 1.4.12 and Proposition 1.4.15-(i) we can apply Theorem 1.6.3 by selecting the

Feller semigroups

$$P^{(\nu)} = P^{\beta}, \quad P^{(\nu),\text{kill}} = P^{\beta,\text{kill}}, \quad P^{(\nu),\infty} = P^{\beta,\infty}, \quad \text{and } P^{\Omega} = P^{\alpha}.$$

Then we denote the Feller triplets obtained from Theorem 1.6.3-(i), Theorem 1.6.3-(ii) and Theorem 1.6.3-(iii) respectively by

$$\begin{aligned}
&\left(P^{\beta,\alpha}, C_{\partial\Omega}([0,T]\times\Omega), (\mathcal{L}_{\beta,\alpha}, \text{Dom}(\mathcal{L}_{\beta,\alpha}))\right), \\
&\left(P^{\beta,\alpha,\text{kill}}, C_{0,\partial\Omega}([0,T]\times\Omega), (\mathcal{L}^{\text{kill}}_{\beta,\alpha}, \text{Dom}(\mathcal{L}^{\text{kill}}_{\beta,\alpha}))\right), \quad \text{and} \\
&\left(P^{\beta,\alpha,\infty}, C_{\infty,\partial\Omega}((-\infty,T]\times\Omega), (\mathcal{L}^{\infty}_{\beta,\alpha}, \text{Dom}(\mathcal{L}^{\infty}_{\beta,\alpha}))\right).
\end{aligned}$$

4.1 Weak solution for the inhomogeneous Caputo EE

Definition of weak solution

Define the distributional operator

$$-D_0^{\beta,*}\varphi(s) := \partial_s I_T^{1-\beta}\varphi(s) + \delta_0(ds) I_T^{1-\beta}\varphi(0),$$

where δ_0 is the delta-measure at 0, and the Riemann-Liouville integral $I_T^{1-\beta}$ is defined as

$$I_T^{1-\beta} f(s) := \int_s^T f(t)\frac{(t-s)^{-\beta}dt}{\Gamma(1-\beta)}, \quad s < T.$$

In the current section only the pairing $\langle\cdot,\cdot\rangle$ is defined as

$$\langle f, g\rangle := \int_0^T \int_\Omega f(t,x)g(t,x)dxdt.$$

Definition 4.1.1. Let $f \in L^{\infty}((0,T)\times\Omega)$ and $\phi_0 \in C_{\partial\Omega}(\Omega)$. A function $u \in L^2((0,T)\times\Omega)$ is said to be a *weak solution to problem* (4.6) if

$$\langle u, (-D_0^{\beta,*} + \Delta_\Omega^{\frac{\alpha}{2}})\varphi\rangle = \langle -f, \varphi\rangle, \quad \text{for every } \varphi \in C_c^{1,2}((0,T)\times\Omega), \tag{4.7}$$

and $u(t) \to \phi_0$ a.e. as $t \downarrow 0$.

The next proposition motivates Definition 4.1.1.

Proposition 4.1.2. Let $\varphi \in C_c^1((0,T))$ and $u \in C([0,T])\cap C^1((0,T))$ such that $u' \in L^1((0,T))$. Then

$$\int_0^T D_0^{\beta} u(t)\varphi(t)dt = -\int_0^T u(t)\left(\partial_t I_T^{1-\beta}\varphi(t)\right)dt - u(0)\partial_t I_T^{1-\beta}\varphi(0).$$

Proof. Using Proposition 1.4.12-(iv), Fubini's Theorem and integration by parts, compute

$$\begin{aligned}\int_0^T D_0^\beta u(t)\varphi(t)dt &= \int_{\mathbb{R}}\int_{\mathbb{R}} u'(s)\frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}\varphi(t)\mathbf{1}_{\{0\le t\le T\}}\mathbf{1}_{\{0\le s\le t\}}dsdt \\ &= \int_{\mathbb{R}} u'(s)\mathbf{1}_{\{0\le s\le T\}}\left(\int_s^T \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}\varphi(t)dt\right)ds \\ &= \int_0^T u'(s)I_T^{1-\beta}\varphi(s)ds \\ &= -\int_0^T u(s)\partial_s I_T^{1-\beta}\varphi(s)ds - u(0)I_T^{1-\beta}\varphi(0).\end{aligned}$$

□

From Proposition 4.1.2 and the identity in (1.12), it is straightforward to prove the following lemma.

Lemma 4.1.3. Let $\varphi\in C_c^{1,2}((0,T)\times\Omega)$ and $u\in C_{\partial\Omega}([0,T]\times\Omega)\cap C^{1,2}((0,T)\times\Omega)$ such that $\partial_t u\in L^1((0,T)\times\Omega)$. Then

$$\langle u,(-D_0^{\beta,*}+\Delta_\Omega^{\frac{\alpha}{2}})\varphi\rangle = \langle(-D_0^\beta+\Delta_\Omega^{\frac{\alpha}{2}})u,\varphi\rangle.$$

Existence of a weak solution

Following the ideas of Chapter 2, we define two auxiliary notions of solution for problem (4.6), starting from the abstract evolution equation

$$\mathcal{L}_{\beta,\alpha}u=-f \text{ on } (0,T]\times\overline{\Omega},\quad u=\phi_0 \text{ on } \{0\}\times\overline{\Omega},\quad u\in \mathrm{Dom}(\mathcal{L}_{\beta,\alpha}). \tag{4.8}$$

Definition 4.1.4. Let $f\in C_{\partial\Omega}([0,T]\times\Omega)$ and $\phi_0\in\mathrm{Dom}(\mathcal{L}_\alpha)$ such that $f(0)=-\mathcal{L}_\alpha\phi_0$. We say that a function $u\in C_{\partial\Omega}([0,T]\times\Omega)$ is a *solution in the domain of the generator to problem* (4.6) if u satisfies (4.8).

The next solution is also defined as in Chapter 2, as a pointwise approximation of solutions in the domain of the generator $\{u_n\}_{n\in\mathbb{N}}$ such that the approximating forcing term $\{f_n\}_{n\in\mathbb{N}}$ satisfies a dominated convergence type of condition.

Definition 4.1.5. Let $f\in B([0,T]\times\overline{\Omega})$ and $\phi_0\in\mathrm{Dom}(\mathcal{L}_\alpha)$. We say that a function $u\in B([0,T]\times\overline{\Omega})$ is a *generalised solution to problem* (4.6) if

$$u=\lim_{n\to\infty}u_n \quad \text{pointwise},$$

where each u_n is the solution in the domain of the generator for a corresponding forcing term $f_n\in C_{\partial\Omega}([0,T]\times\Omega)$ such that

$$f_n\to f \text{ a.e. on } (0,T]\times\Omega,\quad \sup_n\|f_n\|_\infty<\infty,\quad \text{and}\quad f_n(0)=-\mathcal{L}_\alpha\phi_0 \text{ for each } n\in\mathbb{N}.$$

Remark 4.1.6. Any generalised solution must satisfy the boundary conditions $u=0$ on $[0,T]\times\partial\Omega$ and $u=\phi_0$ on $\{0\}\times\overline{\Omega}$.

Lemma 4.1.7. Let $\phi_0 \in \mathrm{Dom}(\mathcal{L}_\alpha)$. Then

(i) If $f + \mathcal{L}_\alpha \phi_0 \in C_{0,\partial\Omega}([0,T] \times \Omega)$, then there exists a unique solution in the domain of the generator to problem (4.6).

(ii) If $f \in B([0,T] \times \overline{\Omega})$, then there exists a unique generalised solution to problem (4.6).

(iii) Both solutions in part (i) and (ii) allow the stochastic representation (4.5).

Proof. This follows by Theorem 2.1.4 by observing that assumption (H0) holds for $\nu(t,r) = -r^{-1-\beta}/\Gamma(-\beta)$, assumptions (H1a), (H1b) and the conditions of Theorem 1.6.3 are all satisfied by Proposition 1.4.12 and Proposition 1.4.15. $\square$

We now show that the dual of $\mathcal{L}_{\beta,\alpha}$ is $(-D_0^{\beta,*} + \Delta_\Omega^{\frac{\alpha}{2}})$.

Lemma 4.1.8. Let $u \in \mathrm{Dom}(\mathcal{L}_{\beta,\alpha})$. Then

$$\langle \mathcal{L}_{\beta,\alpha} u, \varphi \rangle = \langle u, (-D_0^{\beta,*} + \Delta_\Omega^{\frac{\alpha}{2}})\varphi \rangle, \quad \text{for every } \varphi \in C_c^{1,2}((0,T) \times \Omega).$$

Proof. By Theorem 1.6.3-(i) and Proposition 1.4.12-(i) we can pick a sequence

$$\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{Span}\left\{C^1([0,T]) \cdot \mathrm{Dom}(\mathcal{L}_\alpha)\right\},$$

such that $u_n \to u$ and $\mathcal{L}_{\beta,\alpha} u_n \to \mathcal{L}_{\beta,\alpha} u$ in $C_{\partial\Omega}([0,T] \times \Omega)$, with the additional property

$$\mathcal{L}_{\beta,\alpha} u_n = (-D_0^\beta + \mathcal{L}_\alpha) u_n, \quad \text{for every } n \in \mathbb{N}. \tag{4.9}$$

Hence, for every $\varphi \in C_c^{1,2}((0,T) \times \Omega)$, as $n \to \infty$

$$\langle \mathcal{L}_{\beta,\alpha} u, \varphi \rangle \leftarrow \langle \mathcal{L}_{\beta,\alpha} u_n, \varphi \rangle = \langle u_n, (-D_0^{\beta,*} + \Delta_\Omega^{\frac{\alpha}{2}})\varphi \rangle \to \langle u, (-D_0^{\beta,*} + \Delta_\Omega^{\frac{\alpha}{2}})\varphi \rangle,$$

where we use Dominated Convergence Theorem (DCT) for both limits, and for the equality we use the identity (4.9) along with Proposition 4.1.2 and the dual identity in Proposition 1.4.15-(ii). $\square$

We now combine Lemma 4.1.8 with the notion of generalised solution to obtain the main theorem of this section

Theorem 4.1.9. *Let $f \in L^\infty((0,T)\times\Omega)$ and $\phi_0 \in C_{\partial\Omega}(\Omega)$. Then the function $u \in B([0,T]\times\overline{\Omega})$ defined in (4.5) is a weak solution to problem (4.6).*

Proof. Assume for the moment that $\phi_0 \in \mathrm{Dom}(\mathcal{L}_\alpha)$. By the definition of a generalised solution we can take an approximating sequence of forcing terms $\{f_n\}_{n\in\mathbb{N}} \subset C_{\partial\Omega}([0,T] \times \Omega)$ such that $f_n \to f$ a.e., $\sup_n \|f_n\|_\infty < \infty$, and the respective solutions in the domain of the generator $\{u_n\}_{n\in\mathbb{N}}$ satisfy

$$u_n(0) = \phi_0 \text{ for all } n \in \mathbb{N}, \quad u_n \to u \text{ pointwise on } [0,T] \times \Omega, \quad \sup_n \|u_n\|_\infty < \infty,$$

where the last property is an immediate consequence of the stochastic representation (4.5).

Hence, we obtain for every $\varphi \in C_c^{1,2}((0,T)\times\Omega)$, as $n\to\infty$

$$\langle -f,\varphi\rangle \leftarrow \langle -f_n,\varphi\rangle = \langle \mathcal{L}_{\beta,\alpha}u_n,\varphi\rangle = \langle u_n,(-D_0^{\beta,*}+\Delta_\Omega^{\frac{\alpha}{2}})\varphi\rangle \to \langle u,(-D_0^{\beta,*}+\Delta_\Omega^{\frac{\alpha}{2}})\varphi\rangle,$$

where we applied DCT for both limits, the first equality is due to the u_n's being solutions in the domain of the generator, and the second equality holds as a consequence of Lemma 4.1.8. Now, for $\phi_0 \in C_{\partial\Omega}(\Omega)$, let $\{\phi_{0,n}\}_{n\in\mathbb{N}} \subset \mathrm{Dom}(\mathcal{L}_\alpha)$ such that $\phi_{0,n}\to\phi_0$ in $C_{\partial\Omega}(\Omega)$. Let u_n be the generalised solution to problem (4.6) for $f\in B([0,T]\times\overline{\Omega})$ and $\phi_n\in\mathrm{Dom}(\mathcal{L}_\alpha)$, and u defined as in (4.5). Then $u_n\to u$ pointwise and $\sup_n\|u_n\|_\infty<\infty$, which in turn implies by DCT

$$\langle -f,\varphi\rangle = \lim_{n\to\infty}\langle u_n,(-D_0^{\beta,*}+\Delta_\Omega^{\frac{\alpha}{2}})\varphi\rangle = \langle u,(-D_0^{\beta,*}+\Delta_\Omega^{\frac{\alpha}{2}})\varphi\rangle.$$

It is clear that we the result holds for $f\in L^\infty((0,T)\times\Omega)$. Finally, the required convergence of u to the initial condition ϕ_0 follows by the argument in Remark 4.3.3, using the stochastic representation (4.5). □

4.2 Classical solution for the inhomogeneous Caputo EE

Definition 4.2.1. Let $f\in C((0,T]\times\Omega)$ and $\phi_0\in C(\Omega)$. A function $u\in C_{\partial\Omega}([0,T]\times\Omega)\cap C^{1,2}((0,T)\times\Omega)$, such that $|\partial_t u(t,x)|\le Ct^{-\gamma}$, for every $(t,x)\in(0,T]\times\Omega$, for some $\gamma\in(0,1)$, $C>0$, is said to be a *classical solution to problem* (4.6) if u satisfies the identities in (4.6), and for every $x\in\Omega$

$$\lim_{t\downarrow 0}|u(t,x)-\phi_0(x)|=0.$$

In this section the pairing $\langle\cdot,\cdot\rangle$ is defined as

$$\langle f,g\rangle := \int_\Omega f(x)g(x)\,dx.$$

The proof of the main theorem of this section (Theorem 4.2.6), extends the eigenfunction expansion argument in [Chen et al., 2012, Thoerem 5.1], using the next lemma as the key extra ingredient. Define for $\lambda\in\mathbb{R}\backslash\{0\}$ and $f\in C([0,T])$

$$F_\lambda\left[f\right](t) := (-\lambda)^{-1}\int_0^t f(r)\partial_t E_\beta(-\lambda(t-r)^\beta)\,dr,\quad t>0.$$

Lemma 4.2.2. Let $\lambda>0$ and $f\in C([0,T])$. Then

(i)
$$\mathbf{E}\left[\int_0^{\tau_0(t)} e^{-\lambda s}f(-X^{t,\beta}(s))\,ds\right] = F_\lambda\left[f\right](t),\quad t>0.$$

(ii) The bound
$$|F_\lambda\left[f\right](t)| \le \frac{c}{\lambda}\|f\|_\infty,\quad t>0, \tag{4.10}$$

holds, and if $f \in C^1([0,T])$ then

$$|\partial_t F_\lambda [f](t)| \leq \frac{c}{\lambda}\left(\|f'\|_\infty + f(0)\frac{\lambda t^{\beta-1}}{1+\lambda t^\beta}\right), \quad t > 0, \tag{4.11}$$

for some positive constant c.

Proof. (i) Given the second identity in (1.9), it is enough to prove the equivalent identity

$$\mathbf{E}\left[\int_0^{\tau_0(t)} e^{-\lambda s} f(-X^{t,\beta}(s))\,ds\right] + u_0\mathbf{E}\left[e^{-\lambda\tau_0(t)}\right] = F_\lambda [f](t) + u_0 E_\beta(-\lambda t^\beta), \tag{4.12}$$

where u_0 is some constant. We show that the lhs of (4.12) is the unique continuous solution to the Caputo initial value problem solved by the rhs of (4.12). Let $w \in C_0([0,T])$ such that $w' \in C([0,T])$. Then $u(t) := (\lambda - \mathcal{L}_\beta)^{-1}w(t) = \mathbf{E}[\int_0^{\tau_0(t)} e^{-\lambda s} w(-X^{t,\beta}(s))\,ds]$ solves the resolvent equation

$$\mathcal{L}_\beta u = \lambda u - w, \quad u(0) = 0,$$

and $u \in \mathrm{Dom}(\mathcal{L}_\beta)$, by Proposition 1.4.12-(i). By the following computation

$$\begin{aligned}\partial_t u(t) &= \partial_t \int_0^t w(t-y)\left(\int_0^\infty e^{-\lambda s} p_s^\beta(y)\,ds\right)dy \\ &= w(0)\int_0^\infty e^{-\lambda s} p_s^\beta(t)\,ds + \int_0^t w'(t-y)\int_0^\infty e^{-\lambda s}p_s^\beta(y)\,ds\,dy, \qquad t>0,\end{aligned}$$

it follows that $u \in C_0^1([0,T])$, and so $\mathcal{L}_\beta u = -D_0^\beta u$ by Proposition 1.4.12-(i). Let $u_0 \in \mathbb{R}$. Then $\bar{u} := u + u_0$ is a continuous solution to the Caputo initial value problem

$$-D_0^\beta \bar{u} = \mathcal{L}_\beta u - D_0^\beta u_0 = \lambda u - w = \lambda\bar{u} - (w + \lambda u_0),$$

with initial value $\bar{u}(0) = u_0$. By [Diethelm, 2010, Theorem 6.5 and Theorem 7.2] we obtain

$$\bar{u} = \text{rhs of (4.12)} \quad \text{for } f = w + \lambda u_0.$$

Now compute

$$\begin{aligned}\bar{u}(t) &= \mathbf{E}\left[\int_0^{\tau_0(t)} e^{-\lambda s}\left(w(-X^{t,\beta}(s)) \pm \lambda u_0\right)ds\right] + u_0 \\ &= \mathbf{E}\left[\int_0^{\tau_0(t)} e^{-\lambda s}\left(w(-X^{t,\beta}(s)) + \lambda u_0\right)ds\right] - \lambda u_0\frac{\mathbf{E}\left[e^{-\lambda\tau_0(t)}\right]-1}{-\lambda} + u_0 \\ &= \mathbf{E}\left[\int_0^{\tau_0(t)} e^{-\lambda s}\left(w(-X^{t,\beta}(s)) + \lambda u_0\right)ds\right] + u_0\mathbf{E}\left[e^{-\lambda\tau_0(t)}\right].\end{aligned}$$

Now, for an arbitrary $f \in C^1([0,T])$, by picking $w \equiv f - f(0)$ and $u_0 \equiv f(0)\lambda^{-1}$, we obtain the equality (4.12). A straightforward application of DCT proves the claim for $f \in C([0,T])$.

(ii) Recall that there exists a constant $c > 0$ such that $0 \leq -\partial_t E_\beta(-\lambda t^\beta) \leq c\frac{\lambda t^{\beta-1}}{1+\lambda t^\beta}$ by

[Diethelm, 2010, Theorem 7.3] and [Krägeloh, 2003, Equation (17)], and $E_\beta(-\lambda t^\beta) \le \frac{c}{1+\lambda t^\beta}$. Then

$$\left|(-\lambda)^{-1}\int_0^t f(r)\partial_t E_\beta(-\lambda(t-r)^\beta)\,dr\right| \le \|f\|_\infty \frac{1-E_\beta(-\lambda t^\beta)}{\lambda} \le \|f\|_\infty \frac{1+c}{\lambda}.$$

For the second inequality we exploit the smoothness of f, computing for $t>0$

$$\begin{aligned}
\partial_t F_\lambda\,[f]\,(t) &= (-\lambda)^{-1}\partial_t\left(-\int_0^t f(r)\partial_r E_\beta(-\lambda(t-r)^\beta)\,dr\right)\\
&= (-\lambda)^{-1}\partial_t\left(\int_0^t f'(r)E_\beta(-\lambda(t-r)^\beta)\,dr - f(t) + f(0)E_\beta(-\lambda t^\beta)\right)\\
&= (-\lambda)^{-1}\left(\int_0^t f'(r)\partial_t E_\beta(-\lambda(t-r)^\beta)\,dr \pm f'(t) + f(0)\partial_t E_\beta(-\lambda t^\beta)\right)\\
&= F_\lambda\left[f'\right](t) - \lambda^{-1} f(0)\partial_t E_\beta(-\lambda t^\beta).
\end{aligned}$$

Then

$$|\partial_t F_\lambda\,[f]\,(t)| \le \|f'\|_\infty \frac{1+c}{\lambda} + f(0)c\frac{t^{\beta-1}}{1+\lambda t^\beta}.$$

$\square$

From the proof of [Chen et al., 2012, Theorem 5.1], we infer the following lemma.

Lemma 4.2.3. Working with the notation of Proposition 1.4.15-(iii):

(i) the system of eigenvectors $\{\psi_n\}_{n\in\mathbb{N}}$ forms an orthonormal basis of $\mathrm{Dom}(\mathcal{L}^k_{\alpha,2}) \subset L^2(\Omega)$. The corresponding eigenvalues can be ordered so that $\lambda_n \le \lambda_{n+1}$, and also $\lambda_n \le \tilde{c}_1 n^{\alpha/d}$ for some constant $\tilde{c}_1 > 0$. Also, for any compact subset K of Ω, $j = 0,1,2$, there are constants $c_1 = c_1(K,j,d,\alpha)$ such that

$$|\nabla^j \psi_n(x)| \le c_1 \lambda_n^{(d+2j)/(2\alpha)}, \tag{4.13}$$

where $c_1(K,0,d,\alpha)$ is independent of K.

(ii) Suppose $\phi_0 \in \mathrm{Dom}(\mathcal{L}^k_{\alpha,2})$ for $k > -1 + (3d+4)/(2\alpha)$. Then $N := \sum_{n=1}^\infty \lambda_n^{2k}\langle \phi_0, \psi_n\rangle^2 < \infty$, and the series

$$\sum_{n=1}^\infty E_\beta(-\lambda_n t^\beta)\langle \phi_0,\psi_n\rangle \psi_n(x) = \mathbf{E}\left[\phi_0(X^{x,\alpha}(\tau_0(t)))\mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}}\right],$$

defines a function in $C_{\partial\Omega}([0,T]\times\Omega) \cap C^{1,2}((0,T)\times\Omega)$, with bounds for $j = 1,2$,

$$\begin{aligned}
\sum_{n=1}^\infty \left|E_\beta(-\lambda_n t^\beta)\langle \phi_0,\psi_n\rangle \nabla^j\psi_n(x)\right| &\le (c_2\sqrt{N})t^{-\beta}\sum_{n=1}^\infty \lambda_n^{(d+4)/(2\alpha)-1-k} < \infty, \quad t>0,\\
\sum_{n=1}^\infty \left|\partial_t E_\beta(-\lambda_n t^\beta)\langle \phi_0,\psi_n\rangle \psi_n(x)\right| &\le c_3 t^{\gamma\beta-1}, \qquad x\in\Omega,
\end{aligned}$$

where $c_2 = c_2(K,j,d,\alpha), c_3 = c_3(\Omega,\alpha)$, and $0 \le \gamma \le 1 \wedge (4/(2\alpha)-1)$.

We will assume that the forcing term f in (4.6) belongs to the space of functions

$$C^1([0,T];\mathrm{Dom}(\mathcal{L}^k_{\alpha,2})) := \left\{ f \in C^1_{\partial\Omega}([0,T]\times\bar{\Omega}) : \sup_t \|f(t)\|_{\mathcal{L}^k_{\alpha,2}} + \sup_t \|\partial_t f(t)\|_{\mathcal{L}^k_{\alpha,2}} < \infty \right\}. \tag{4.14}$$

Note that if $f \in C^1([0,T];\mathrm{Dom}(\mathcal{L}^k_{\alpha,2}))$, then there exists $M>0$ such that for every $n\in\mathbb{N}$

$$\sup_{t\in[0,T]} |\langle f(t),\psi_n\rangle| \le M\lambda_n^{-k}, \quad \text{and} \quad \sup_{t\in[0,T]} |\langle \partial_t f(t),\psi_n\rangle| \le M\lambda_n^{-k}. \tag{4.15}$$

Remark 4.2.4. The inclusion $\mathrm{Span}\{C^1([0,T])\cdot \mathrm{Dom}(\mathcal{L}^k_{\alpha,2})\} \subset C^1([0,T];\mathrm{Dom}(\mathcal{L}^k_{\alpha,2}))$ is clear. Moreover, if $k\in\mathbb{N}$, then the inclusion $C^{1,2k}_c([0,T]\times\Omega) \subset C^1([0,T];\mathrm{Dom}(\mathcal{L}^k_{\alpha,2}))$ holds[3]. To see this, let $f\in C^{1,2k}_c([0,T]\times\Omega)$ and compute for each $t\in[0,T]$

$$\sum_{n=1}^\infty \lambda_n^{2k}\langle f(t),\psi_n\rangle^2 = \sum_{n=1}^\infty \langle f(t),\mathcal{L}^k_{\alpha,2}\psi_n\rangle^2 = \sum_{n=1}^\infty \langle (\Delta_\Omega^{\frac{\alpha}{2}})^k f(t),\psi_n\rangle^2 = \|(\Delta_\Omega^{\frac{\alpha}{2}})^k f(t)\|^2_{L^2(\Omega)} < \infty,$$

where the second equality holds by the same argument at the end the proof of Theorem 4.2.6, using $(\Delta_\Omega^{\frac{\alpha}{2}})^m f(t) \in L^2(\Omega)$ for each $t\in[0,T]$ and $m\le k$. Now observe that by DCT the function $t\mapsto \|(\Delta_\Omega^{\frac{\alpha}{2}})^k f(t)\|_{L^2(\Omega)}$ is continuous on $[0,T]$, because $(\Delta_\Omega^{\frac{\alpha}{2}})^k f \in C([0,T]\times\bar{\Omega})$. Repeat the argument for $\partial_t f$ to conclude.

Lemma 4.2.5. If $f(t)\in\mathrm{Dom}(\mathcal{L}^k_{\alpha,2})$ for $k > -1 + (3d+4)/(2\alpha)$, for every $t\in[0,T]$, and $f\in C_{\partial\Omega}([0,T]\times\Omega)$, then

$$\mathbf{E}\left[\int_0^{\tau_{t,x}} f\left(-X^{t,\beta}(s), X^{x,\alpha}(s)\right) ds\right] = \sum_{n=1}^\infty \psi_n(x) F_{\lambda_n}\left[\langle f(\cdot),\psi_n\rangle\right](t).$$

If in addition $f\in C^1([0,T];\mathrm{Dom}(\mathcal{L}^k_{\alpha,2}))$, then there exists a constant C such that for $t\in(0,T]$

$$\sum_{n=1}^\infty |\psi_n(x)\partial_t F_{\lambda_n}\left[\langle f(\cdot),\psi_n\rangle\right](t)| \le Ct^{\beta-1}. \tag{4.16}$$

Proof. We justify the following equalities

$$\begin{aligned}
\mathbf{E}\left[\int_0^{\tau_{t,x}} f\left(-X^{t,\beta}(s), X^{x,\alpha}(s)\right) ds\right] &= \int_0^\infty P_s^{\beta,\mathrm{kill}} P_s^\Omega f(t,x)\, ds \\
&= \int_0^\infty P_s^{\beta,\mathrm{kill}} \left(\sum_{n=1}^\infty \langle f(t),\psi_n\rangle \psi_n(x) e^{-s\lambda_n}\right) ds \\
&= \sum_{n=1}^\infty \psi_n(x) \int_0^\infty P_s^{\beta,\mathrm{kill}} \langle f(t),\psi_n\rangle e^{-s\lambda_n}\, ds \\
&= \sum_{n=1}^\infty \psi_n(x) \mathbf{E}\left[\int_0^{\tau_0(t)} \langle f(-X^{t,\beta}(s)),\psi_n\rangle e^{-s\lambda_n}\, ds\right] \\
&= \sum_{n=1}^\infty \psi_n(x) F_{\lambda_n}\left[\langle f(\cdot),\psi_n\rangle\right](t).
\end{aligned}$$

[3]We define $C^{1,2k}_c([0,T]\times\Omega) = C^{1,2k}((0,T)\times\Omega) \cap \{f,\partial_t f \in C([0,T]\times\Omega), \mathrm{supp}\{f\}\subset[0,T]\times\Omega \text{ is compact}\}$.

We can apply Fubini's Theorem in the third equality as

$$\sum_{n=1}^{\infty} |\langle f(t), \psi_n\rangle| \|\psi_n\|_\infty \leq C \sum_{n=1}^{\infty} n^{(\alpha/d)(d/(2\alpha)-k)} < \infty,$$

for some constant $C > 0$, each $t \geq 0$ and any $k > 3d/(2\alpha)$, using the bounds in Lemma 4.2.3-(i) and in (4.15). We apply Lemma 4.2.2-(i) in the fifth equality as $r \mapsto \langle f(r), \psi_n\rangle \in C([0,T])$ for each $n \in \mathbb{N}$. The other equalities are clear.
For the last claim we use the bounds in (4.11), (4.15) and Lemma 4.2.3-(i) to obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} |\psi_n(x) \partial_t F_{\lambda_n} [\langle f(t), \psi_n\rangle](t)| &\leq \sum_{n=1}^{\infty} |\psi_n(x)| \frac{c}{\lambda_n} \left(\sup_{r\in[0,T]} |\langle \partial_r f(r), \psi_n\rangle| + \frac{\lambda_n t^{\beta-1}}{1+\lambda_n t^\beta} |\langle f(0), \psi_n\rangle| \right) \\
&\leq \sum_{n=1}^{\infty} |\psi_n(x)| \frac{cM\lambda_n^{-k}}{\lambda_n} \left(1 + \frac{\lambda_n t^{\beta-1}}{1+\lambda_n t^\beta}\right) \\
&\leq (c_1 cM) \sum_{n=1}^{\infty} \frac{\lambda_n^{d/(2\alpha)} \lambda_n^{-k}}{\lambda_n} \left(1 + \frac{\lambda_n t^{\beta-1}}{1+\lambda_n t^\beta}\right) \\
&\leq (c_1 cM) t^{\beta-1} \sum_{n=1}^{\infty} \lambda_n^{d/(2\alpha)-k} \\
&\leq (\tilde{c}_1 c_1 cM) t^{\beta-1} \sum_{n=1}^{\infty} n^{(\alpha/d)(d/(2\alpha)-k)} < \infty,
\end{aligned}$$

for any $k > 3d/(2\alpha)$, where the constants $\tilde{c}_1, c_1, c$ and M follow the notation of the referenced inequalities, and a constant is omitted in the fourth inequality. □

Theorem 4.2.6. *Let $\Omega \subset \mathbb{R}^d$ be a regular set. Assume that $\phi_0 \in Dom(\mathcal{L}_{\alpha,2}^k)$, and $f \in C^1([0,T]; Dom(\mathcal{L}_{\alpha,2}^k))$ for some $k > -1 + (3d+4)/(2\alpha)$, where $C^1([0,T]; Dom(\mathcal{L}_{\alpha,2}^k))$ is defined in (4.14). Then*

$$\begin{aligned}
&u \in C_{\partial\Omega}([0,T] \times \Omega) \cap C^{1,2}((0,T) \times \Omega), \quad \text{and} \\
&|\partial_t u(t,x)| \leq Ct^{-\gamma}, \ \text{for every } (t,x) \in (0,T] \times \Omega, \ \text{for some } \gamma \in (0,1), \ C > 0,
\end{aligned} \tag{4.17}$$

where u is defined in (4.5). Moreover, u is the unique classical solution to problem (4.6).

Proof. (The notation for constants is consistent with the referenced inequalities.)
By Lemma 4.2.3-(ii) and Lemma 4.2.5 we can write our candidate solution (4.5) as

$$u(t,x) = \sum_{n=1}^{\infty} E_\beta(-\lambda_n t^\beta) \langle \phi_0, \psi_n\rangle \psi_n(x) + \sum_{n=1}^{\infty} F_{\lambda_n} [\langle f(\cdot), \psi_n\rangle](t) \psi_n(x),$$

and the first series enjoys the regularity properties stated in (4.17). We now prove the same regularity for the second series. Observe that $\sum_{n=1}^{\infty} F_{\lambda_n} [\langle f(\cdot), \psi_n\rangle](t)\psi_n(x)$ converges uniformly to a function in $C_{\partial\Omega}([0,T]\times\Omega)$, since we have the uniform bound

$$\begin{aligned}\sum_{n=1}^{\infty}\left|F_{\lambda_n}\left[\langle f(\cdot),\psi_n\rangle\right](t)\psi_n(x)\right| &\leq \sum_{n=1}^{\infty} c\lambda_n^{-1}\|\langle f(\cdot),\psi_n\rangle\|_{C([0,T])}c_1\lambda_n^{d/(2\alpha)}\\ &\leq (cc_1M)\sum_{n=1}^{\infty}\lambda_n^{-1-k+d/(2\alpha)}\\ &\leq (\tilde{c}_1c_1cM)\sum_{n=1}^{\infty}n^{(\alpha/d)(d/(2\alpha)-k-1)}<\infty,\end{aligned}$$

for any $k>-1+3d/(2\alpha)$, using the bounds in (4.15), (4.10) and Lemma 4.2.3-(i). Further, for $j=1,2$, and for any x in a compact subset K of Ω, the term-wise space derivative of u can be bounded as follows,

$$\begin{aligned}\sum_{n=1}^{\infty}\left|F_{\lambda_n}\left[\langle f(\cdot),\psi_n\rangle\right](t)\right|\|\nabla^j\psi_n\|_{\infty} &\leq \sum_{n=1}^{\infty} c\lambda_n^{-1}\|\langle f(\cdot),\psi_n\rangle\|_{C([0,T])}c_1\lambda_n^{(d+4)/2\alpha}\\ &\leq (\tilde{c}_1c_1cM)\sum_{n=1}^{\infty}n^{(\alpha/d)((d+4)/(2\alpha)-k-1)}<\infty,\end{aligned}\tag{4.18}$$

as

$$\frac{\alpha}{d}\left(\frac{d+4}{2\alpha}-k-1\right)<-1 \iff k>\frac{3d+4-2\alpha}{2\alpha},$$

where we use the bounds in (4.15), (4.10) and Lemma 4.2.3-(i). Thus, Weierstrass M-test implies that for any $t>0$, $u(t)$ is a C^2 function on every $K\subset\Omega$ compact. For the time regularity we use the inequality (4.16) from Lemma 4.2.5[4].
By Theorem 4.1.9, u is also a weak solution to problem (4.6), and by Lemma 4.1.3 and standard approximation arguments, u satisfies the equalities in (4.6). Continuity at $t=0$ can be proved as in Remark 4.3.3.

To prove uniqueness, consider two classical solutions to problem (4.6), denoted by u,v. Then $w:=u-v$ is a classical solution to problem (4.6) with $f=0$, $\phi_0=0$. Consider the continuous functions on $[0,T]$, $t\mapsto\langle w(t),\psi_n\rangle$, $n\in\mathbb{N}$. If we can justify

$$D_0^{\beta}\langle w(t),\psi_n\rangle=\langle D_0^{\beta}w(t),\psi_n\rangle=\langle\Delta_{\Omega}^{\frac{\alpha}{2}}w(t),\psi_n\rangle=\langle w(t),\mathcal{L}_{\alpha,2}\psi_n\rangle=-\lambda_n\langle w(t),\psi_n\rangle,\tag{4.19}$$

for $t>0$, it follows by [Diethelm, 2010, Theorem 6.5 and Theorem 7.2] that $\langle w(t),\psi_n\rangle=0$ for every $t\in[0,T]$, $n\in\mathbb{N}$, and we are done. The first equality is a consequence of $|\partial_r w(r,y)|\leq Cr^{-\gamma}$, for some $\gamma\in(0,1)$. The second and fourth equalities in (4.19) are clear. Now, as $\psi_n\in\mathrm{Dom}(\mathcal{L}_{\alpha,2})$, there exists a sequence $\{\psi_{n,j}\}_{j\in\mathbb{N}}\subset C_c^{\infty}(\Omega)$, such that as $j\to\infty$

$$\psi_{n,j}\to\psi_n,\quad\text{and}\quad \Delta_{\Omega}^{\frac{\alpha}{2}}\psi_{n,j}=\mathcal{L}_{\alpha,2}\psi_{n,j}\to\mathcal{L}_{\alpha,2}\psi_n,\quad\text{in } L^2(\Omega),\tag{4.20}$$

where the equality in (4.20) holds by [Chen et al., 2012, Lemma 4.1]. Combining (4.20) with the equality (1.12) and $\Delta_{\Omega}^{\frac{\alpha}{2}}w(t)\in L^2(\Omega)$ for each $t>0$, the third equality in (4.19) is proven. $\square$

[4] From the proof of Lemma 4.2.5 it follows that if $\phi_0=f(0)=0$, then $\partial_t u$ is bounded.

4.3 Classical solution for the Marchaud EE

4.3.1 Stochastic representation and continuity at $t = 0$

Lemma 4.3.1. Define the function $f_\phi : (0, T] \times \Omega \to \mathbb{R}$ as

$$f_\phi(t, x) := \int_t^\infty (\phi(t - r, x) - \phi(t, x)) \frac{-\Gamma(-\beta)^{-1} dr}{r^{1+\beta}}, \tag{4.21}$$

assuming that $\phi \in C_{\infty,\partial\Omega}((-\infty, 0] \times \Omega)$, $\phi(0) \in \mathrm{Dom}(\mathcal{L}_\alpha)$, and the extension of ϕ to $\phi(0)$ on $(0, T] \times \overline{\Omega}$ is such that

$$\phi \in \mathrm{Dom}(\mathcal{L}^\infty_{\beta,\alpha}), \quad \text{and} \quad \mathcal{L}^\infty_{\beta,\alpha}\phi = (-D^\beta_\infty + \mathcal{L}_\alpha)\phi. \tag{4.22}$$

Then $f_\phi \in C([0, T] \times \overline{\Omega})$ and the function u defined in (4.5) for $f = f_\phi$ and $\phi_0 = \phi(0)$, equals the function $\tilde{u}$ defined in (4.3) for $g = 0$, on $(0, T] \times \Omega$.

Proof. The first claim follows from $f_\phi = -D^\beta_\infty\phi \in C([0, T] \times \overline{\Omega})$, using (4.22) and $\mathcal{L}_\alpha\phi(t, x) = \mathcal{L}_\alpha\phi(0, x)$ for all $(t, x) \in [0, T] \times \overline{\Omega}$. Recall that we write $\tau_{t,x} = \tau_0(t) \wedge \tau_\Omega(x)$. Fix $(t, x) \in (0, T] \times \Omega$. It is enough to justify the following equalities

$$\begin{aligned}
u(t, x) &= \mathbf{E}\left[\phi(0, X^{x,\alpha}(\tau_0(t))\mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}} + \int_0^{\tau_{t,x}} f_\phi\left(-X^{t,\beta}(s), X^{x,\alpha}(s)\right) ds\right] \\
&= \mathbf{E}\left[\phi(0, x) + \int_0^{\tau_{t,x}} \mathcal{L}_\alpha\phi\left(0, X^{x,\alpha}(s)\right) ds + \int_0^{\tau_{t,x}} f_\phi\left(-X^{t,\beta}(s), X^{x,\alpha}(s)\right) ds\right] \\
&= \mathbf{E}\left[\int_0^{\tau_{t,x}} \mathcal{L}_\alpha\phi\left(-X^{t,\beta}(s), X^{x,\alpha}(s)\right) - D^\beta_\infty\phi\left(-X^{t,\beta}(s), X^{x,\alpha}(s)\right) ds\right] + \phi(0, x) \\
&= \mathbf{E}\left[\int_0^{\tau_{t,x}} \mathcal{L}^\infty_{\beta,\alpha}\phi\left(-X^{t,\beta}(s), X^{x,\alpha}(s)\right) ds\right] + \phi(0, x) \\
&= \mathbf{E}\left[\phi\left(-X^{t,\beta}(\tau_{t,x}), X^{x,\alpha}(\tau_{t,x})\right)\right] \pm \phi(0, x).
\end{aligned}$$

For the second equality we use Dynkin formula with Theorem 1.6.3-(i) and $\phi(0) \in \mathrm{Dom}(\mathcal{L}_\alpha)$; for the third equality, as we extended $\phi(t, x) = \phi(0, x)$ on $[0, T] \times \Omega$, we use the identities $f_\phi(t, x) = -D^\beta_\infty\phi(t, x)$ and $\mathcal{L}_\alpha\phi(0, x) = \mathcal{L}_\alpha\phi(t, x)$ on $(0, T] \times \Omega$; in the fourth equality we use assumption (4.22); the fifth equality is again an application of Dynkin formula with Theorem 1.6.3-(iii) and $\phi(t, x) = \phi(0, x)$ on $(0, T] \times \Omega$. □

Corollary 4.3.2. If $\phi \in C^1_{b,\partial\Omega}((-\infty, 0] \times \Omega)$, then for $(t, x) \in (0, T] \times \Omega$

$$\mathbf{E}\left[\phi\big(0, X^{x,\alpha}(\tau_{t,x})\big) + \int_0^{\tau_{t,x}} f_\phi\left(-X^{t,\beta}(s), X^{x,\alpha}(s)\right) ds\right] = \mathbf{E}\left[\phi\left(-X^{t,\beta}(\tau_{t,x}), X^{x,\alpha}(\tau_{t,x})\right)\right]. \tag{4.23}$$

Proof. *Step 1.* We prove (4.23) for $\phi \in C^1_{\infty,\partial\Omega}((-\infty, 0] \times \Omega) \cap \{\partial_t f(0) = 0\}$ with compact support in $(-\infty, 0] \times \overline{\Omega}$. For such ϕ, let $K > 0$ such that ϕ is supported in $(-K, 0] \times \overline{\Omega}$. By the same arguments as in the proof of Theorem 1.6.3-(ii), it follows that $\mathrm{Span}\{C([-K, 0]) \cap \{f(-K) = f(0) = 0\} \cdot C_{\partial\Omega}(\Omega)\}$ is dense in $C_{\partial\Omega}([-K, 0] \times \Omega) \cap \{f(-K) = f(0) = 0\}$ with respect to the supremum norm. We can use this fact to construct a sequence $\{\phi_n\}_{n\in\mathbb{N}} \in \mathrm{Span}\{C^1_\infty(-\infty, 0]) \cap$

$\{f'(0)=0\}\cdot C_{\partial\Omega}(\Omega)\}$ such that

$$\|\phi_n-\phi\|_{C((-\infty,0]\times\bar{\Omega})}+\|\partial_t(\phi_n-\phi)\|_{C((-\infty,0]\times\bar{\Omega})}\to 0,\quad \text{as } n\to\infty.$$

Moreover, it follows that $f_{\phi_n}\to f_\phi$ as $n\to\infty$ pointwise on $[0,T]\times\Omega$ and $\sup_n\|f_{\phi_n}\|_{C([0,T]\times\bar{\Omega})}$ is finite. It remains to show that (4.23) holds for functions in $\text{Span}\{C^1_\infty(-\infty,0])\cap\{f'(0)=0\}\cdot C_{\partial\Omega}(\Omega)\}$, as DCT applied to the sequences above yields the claim. By Theorem 1.6.3-(iii) with $C^\infty_{(\nu)}=C^1_\infty((-\infty,T])$, Proposition 1.4.12 and Lemma 4.3.1, equality (4.23) holds for $\phi\in\text{Span}\{C^1_\infty((-\infty,0])\cap\{f'(0)=0\}\cdot\text{Dom}(\mathcal{L}_\alpha))\}$. As $\text{Dom}(\mathcal{L}_\alpha)$ is dense in $C_{\partial\Omega}(\Omega)$, equality (4.23) holds for $\phi\in\text{Span}\{C^1_\infty((-\infty,0])\cap\{f'(0)=0\}\cdot C_{\partial\Omega}(\Omega)\}$ by DCT.

Step 2. For $\phi\in C^1_{b,\partial\Omega}((-\infty,0]\times\Omega)$, take a sequence $\{\phi_n\}_{n\in\mathbb{N}}\subset C^1_{\infty,\partial\Omega}((-\infty,0]\times\Omega)\cap\{\partial_t f(0)=0\}$ compactly supported in $(-\infty,0]\times\bar{\Omega}$, such that $\phi_n\to\phi$ pointwise on $(-\infty,0]\times\Omega$, and $\sup_n\|\phi_n\|_{C((-\infty,0]\times\bar{\Omega})}+\sup_n\|\partial_t\phi_n\|_{C((-\infty,0]\times\bar{\Omega})}<\infty$. Then $f_{\phi_n}\to f_\phi$ pointwise on $[0,T]\times\Omega$ and $\sup_n\|f_{\phi_n}\|_{C([0,T]\times\bar{\Omega})}<\infty$. Finally, apply DCT to both sides of (4.23). □

Remark 4.3.3. If we can apply Corollary 4.3.2, then we can prove continuity at $t=0$ for the solution (4.3) via the following argument

$$\begin{aligned}|\text{Formula (4.5)}-\phi_0(x)| &\le |\mathbf{E}\left[\phi_0\left(X^{x,\alpha}(\tau_0(t)\wedge\tau_\Omega(x))\right)-\phi_0(x)\right]|+\|f\|_\infty\mathbf{E}\left[\tau_0(t)\right]\\ &= o_{t\downarrow 0}(1)\ +\|f\|_\infty\frac{t^\beta}{\Gamma(\beta+1)},\end{aligned}$$

for each $x\in\Omega$, using stochastic continuity of the process[5] $t\mapsto X^{x,\alpha}(\tau_0(t))$ at $t=0$. One could also use stochastic continuity at $t=0$ of $-X^{t,\beta}(\tau_0(t))=t-X^\beta(\tau_0(t))$, bypassing Corollary 4.3.2. In Proposition 4.4.2 in the Appendix we prove continuity at $t=0$ by proving a bound on big overshootings $-X^{t,\beta}(\tau_0(t))$ for small times.

4.3.2 Equivalence of the classical solutions to problems (4.1) and (4.6)

Definition 4.3.4. Let $\phi\in C_{b,\partial\Omega}((-\infty,0]\times\Omega)$ and $g\in C((0,T]\times\Omega)$. A function $\tilde{u}\in C_{b,\partial\Omega}((-\infty,T]\times\Omega)\cap C^{1,2}((0,T)\times\Omega)$ such that $|\partial_t\tilde{u}(t,x)|\le Ct^{-\gamma}$, for every $(t,x)\in(0,T]\times\Omega$, for some $\gamma\in(0,1)$, $C>0$, is said to be a *classical solution to problem* (4.1) if $\tilde{u}$ satisfies the identities in (4.1), and for every $x\in\Omega$

$$\lim_{t\downarrow 0}|\tilde{u}(t,x)-\phi(0,x)|=0.$$

Lemma 4.3.5. Let $\phi\in C_{b,\partial\Omega}((-\infty,0]\times\Omega)$ such that $f_\phi\in C((0,T]\times\Omega)$, where f_ϕ is defined in (4.21), and let $g\in C((0,T]\times\Omega)$. Then, if u is a classical solution to problem (4.6) with $f=f_\phi+g$ and $\phi_0=\phi(0)$, then the extension

$$\tilde{u}:=\begin{cases}u, & \text{in } (0,T]\times\bar{\Omega},\\ \phi, & \text{in } (-\infty,0]\times\Omega,\end{cases}$$

is a classical solution to problem (4.1). Conversely, if $\tilde{u}$ is a classical solution to problem (4.1),

[5]This follows as $X^{x,\alpha}(s)$ is right continuous and $\tau_0(t)$ is right continuous, non-decreasing with $\tau_0(0)=0$.

then the restriction of $\tilde{u}$ to $[0,T]\times\overline{\Omega}$ is a classical solution to problem (4.6) with $f=f_\phi+g$ and $\phi_0=\phi(0)$.

Proof. The equivalence of convergence to initial data and the required regularities are clear. It is also immediate that $\Delta_\Omega^{\frac{\alpha}{2}}u=\Delta_\Omega^{\frac{\alpha}{2}}\tilde{u}$ on $(0,T]\times\Omega$. Write $\nu(r)=-\Gamma(-\beta)^{-1}r^{-1-\beta}$. On $(0,T]\times\Omega$ we have the equality

$$\begin{aligned}
-D_\infty^\beta\tilde{u}(t,x) &= \int_0^\infty(\tilde{u}(t-r,x)-\tilde{u}(t,x))\,\nu(r)dr\\
&= \int_0^t(\tilde{u}(t-r,x)-\tilde{u}(t,x))\,\nu(r)dr+\int_t^\infty\phi(t-r,x)\,\nu(r)dr\\
&\qquad -\tilde{u}(t,x))\int_t^\infty\nu(r)dr\pm\phi(0,x)\int_t^\infty\nu(r)dr\\
&= -\,D_0^\beta\tilde{u}(t,x)+f_\phi(t,x).
\end{aligned}$$

This is enough to prove both directions. ☐

4.3.3 Main result

Theorem 4.3.6. *Let $\Omega\subset\mathbb{R}^d$ be a regular set. Assume that $\phi\in C^1_{b,\partial\Omega}((-\infty,0]\times\Omega)$ with $\phi(0)\in Dom(\mathcal{L}^k_{\alpha,2})$ and $f_\phi,g\in C^1([0,T];Dom(\mathcal{L}^k_{\alpha,2}))$, for some $k>-1+(3d+4)/(2\alpha)$, where f_ϕ is defined in (4.21) and $C^1([0,T];Dom(\mathcal{L}^k_{\alpha,2}))$ is defined in (4.14). Then*

$$\begin{aligned}
&\tilde{u}\in C_{b,\partial\Omega}((-\infty,T]\times\Omega)\cap C^{1,2}((0,T)\times\Omega),\quad \textit{and}\\
&|\partial_t\tilde{u}(t,x)|\le Ct^{-\gamma},\ \textit{for every } (t,x)\in(0,T]\times\Omega,\ \textit{for some } \gamma\in(0,1),\ C>0,
\end{aligned}$$

where $\tilde{u}$ is defined as in (4.3). Moreover, $\tilde{u}$ is the unique classical solution to problem (4.1).

Proof. By the assumptions on ϕ and g, and Lemma 4.3.5, existence and uniqueness of classical solutions follows by Theorem 4.2.6 with $\phi_0=\phi(0)$ and $f=f_\phi+g$. Now apply Corollary 4.3.2 to obtain the stochastic representation (4.3) from the stochastic representation (4.5). ☐

Remark 4.3.7. Using the Proposition 3.1.6 in Chapter 3 (or [Ikeda and Watanabe, 1962, Theorem 1 for $\lambda=0$]), given $\mathbf{P}[-X^t(\tau_0(t))\in\{0\}]=0$ for every $t>0$ (see [Bertoin, 1996, III, Theorem 4]) and the independence of $X^{x,\alpha}$ and $-X^{t,\beta}$, we obtain for $(t,x)\in(0,T]\times\Omega$

$$\mathbf{E}\left[\phi\left(-X^{t,\beta}(\tau_0(t)),X^{x,\alpha}(\tau_0(t))\right)\mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}}\right]=\int_{-\infty}^0\int_\Omega\phi(r,y)H^{t,x}_{\beta,\alpha}(r,y)\,dr\,dy,$$

with the heat kernel

$$H^{t,x}_{\beta,\alpha}(r,y)=\int_0^t\frac{-\Gamma(-\beta)^{-1}}{(z-r)^{1+\beta}}\left(\int_0^\infty p_s^\Omega(x,y)p_s^\beta(t-z)\,ds\right)dz.$$

It is straightforward to compute for $(t,x)\in(0,T]\times\Omega$

$$\mathbf{E}\left[\int_0^{\tau_0(t)\wedge\tau_\Omega(x)}g\left(-X^{t,\beta}(s),X^{x,\alpha}(s)\right)ds\right]=\int_0^t\int_\Omega g(z,y)\left(\int_0^\infty p_s^\Omega(x,y)p_s^\beta(t-z)\,ds\right)dz\,dy.$$

Remark 4.3.8. Notice that the value $\phi(0)$ does not contribute to the solution (4.3) because $\mathbf{P}[-X^t(\tau_0(t)) \in \{0\}] = 0$ for all $t > 0$. However, $u(t) \to \phi(0)$ as $t \downarrow 0$. We discuss the continuity of the solution at $t = 0$ in more detail in Appendix 4.4.

Remark 4.3.9. We could drop the condition $\|\partial_t \phi\|_\infty < \infty$ in Theorem 4.3.6, by weakening Corollary 4.3.2, for example to ϕ being β^*-Hölder continuous at $t = 0$, for some $\beta^* > \beta$ and $\phi \in L^\infty((-\infty, 0) \times \Omega)$. This is essentially because $\lim_{t\downarrow 0} f_\phi(t)$ remains well-defined. However, in order to apply Theorem 4.2.6 in the proof of Theorem 4.3.6 we need to assume $f_\phi \in C^1([0,T]; \mathrm{Dom}(\mathcal{L}^k_{\alpha,2}))$. Hence, a minimal requirement is that ϕ is continuously differentiable in time and both ϕ and $\partial_t\phi$ are $\mathcal{O}(|r|^{\beta_*})$ at $-\infty$ and β^*-Hölder continuous at 0, for some $\beta_* < \beta < \beta^*$, as we need f_ϕ and $\partial_t f_\phi$ to be continuous on $[0,T] \times \overline{\Omega}$.

Remark 4.3.10. Suppose that $\phi \in C^{2,2k}_{\infty,\partial\Omega}((-\infty, 0] \times \Omega)$ and $\phi(t)$ along with its partial derivatives in space are compactly supported in Ω, for each $t \in (-\infty, 0]$, where $k \in \mathbb{N}$ and $k > -1+(3d+4)/(2\alpha)$. Then, an application of Remark 4.2.4 implies that $f_\phi \in C^1([0,T]; \mathrm{Dom}(\mathcal{L}^k_{\alpha,2}))$.

4.4 Remarks on convergence to the initial condition $\phi(0)$

Proposition 4.4.1. For every $p, \varepsilon > 0$, the following bound on small overshootings holds,

$$\mathbf{P}[X^{t,\beta}(\tau_0(t)) \leq \varepsilon] \geq (1-p), \quad \text{for every } t \leq \varepsilon p^{\frac{1}{\beta}}.$$

Proof. With the first equality holding by [Ikeda and Watanabe, 1962, Theorem 1 for $\lambda = 0$] along with the identity (1.10), compute

$$\begin{aligned}
\mathbf{P}[X^{t,\beta}(\tau_0(t)) \leq \varepsilon] &= \int_{-\varepsilon}^0 \left(\frac{1}{\Gamma(\beta)} \int_0^t (-\partial_y (y-r)^{-\beta}) \frac{(t-y)^{\beta-1}}{\Gamma(1-\beta)}\, dy \right) dr \\
&= \int_{-\varepsilon}^0 \left(\frac{\beta}{\Gamma(\beta)\Gamma(1-\beta)} \int_0^t (y-r)^{-\beta-1}(t-y)^{\beta-1}\, dy \right) dr \\
&= \frac{-\Gamma(\beta)^{-1}}{\Gamma(-\beta)} \int_0^t (t-y)^{\beta-1} \left(\int_{-\varepsilon}^0 (y-r)^{-\beta-1} dr \right) dy \\
&= \frac{-\Gamma(\beta)^{-1}\beta^{-1}}{\Gamma(-\beta)} (a - a_\varepsilon(t)),
\end{aligned}$$

where $a_\varepsilon(t) := \int_0^t (t-y)^{\beta-1}(y+\varepsilon)^{-\beta}\, dy$ and $a := \int_0^t (t-y)^{\beta-1} y^{-\beta}\, dy = \Gamma(\beta)\Gamma(1-\beta)$ for every $t > 0$. Now pick $\tilde{t} = \varepsilon p^{1/\beta}$. Then for every $0 \leq y \leq \tilde{t}$

$$(y+\varepsilon)^{-\beta} = (y + p^{-1/\beta}\tilde{t})^{-\beta} \leq p\tilde{t}^{-\beta} \leq p y^{-\beta},$$

hence for every $t \leq \tilde{t}$

$$\frac{a_\varepsilon(t)}{a} = \frac{\int_0^t (t-y)^{\beta-1}(y+\varepsilon)^{-\beta} dy}{\int_0^t (t-y)^{\beta-1} y^{-\beta}\, dy} \leq p.$$

Then $a_\varepsilon(t) \leq pa$ for every $t \leq \tilde{t}$, which is equivalent to $a - a_\varepsilon(t) \geq (1-p)a$ for every $t \leq \tilde{t}$. And so we obtain

$$\mathbf{P}[X^{t,\beta}(\tau_0(t)) \le \varepsilon] \ge (1-p)\frac{-\Gamma(\beta)^{-1}}{\Gamma(-\beta)}\beta^{-1}\Gamma(\beta)\Gamma(1-\beta) = (1-p).$$

□

We now use the bound in Proposition 4.4.1 to prove the following continuity result

Proposition 4.4.2. Consider the function $\tilde{u}$ defined in (4.3), with an arbitrary Ω-valued stochastic (sub-)process X^x in place of $X^{x,\alpha}$, such that $t \mapsto X^x(\tau_0(t))$ is stochastically continuous at $t = 0$. Also assume $\phi \in B((-\infty, 0] \times \Omega))$ and ϕ is continuous at every point in $\{0\} \times \Omega$. Then for every $x \in \Omega$

$$\lim_{t \downarrow 0} |\tilde{u}(t,x) - \phi(0,x)| = 0.$$

Proof. Let $x \in \Omega$. Let $\delta > 0$ be arbitrary. Pick $\varepsilon, \varepsilon' > 0$ such that

$$\sup_{(s,y)\in(-\varepsilon,0]\times B_{\varepsilon'}(x)} |\phi(s,y) - \phi(0,x)| \le \delta.$$

Then

$$\begin{aligned}
|\tilde{u}(t,x) - \phi(0,x)| &\le \left|\mathbf{E}\left[(\phi(-X^{t,\beta}(\tau_0(t)), X^x(\tau_0(t)) - \phi(0,x))\mathbf{1}_{\{X^{t,\beta}(\tau_0(t))>\varepsilon\}}\right]\right| \\
&\quad + \left|\mathbf{E}\left[(\phi(-X^{t,\beta}(\tau_0(t)), X^x(\tau_0(t))) - \phi(0,x))\mathbf{1}_{\{X^{t,\beta}(\tau_0(t))\le\varepsilon\}}\right]\right| \\
&\le 2\|\phi\|_\infty \mathbf{P}[X^{t,\beta}(\tau_0(t)) > \varepsilon] \\
&\quad + \mathbf{E}\left[|\phi(-X^{t,\beta}(\tau_0(t)), X^x(\tau_0(t))) - \phi(0,x)|\mathbf{1}_{\{X^{t,\beta}(\tau_0(t))\le\varepsilon, |X^x(\tau_0(t)))-x|\le\varepsilon'\}}\right] \\
&\quad + \mathbf{E}\left[|\phi(-X^{t,\beta}(\tau_0(t)), X^x(\tau_0(t))) - \phi(0,x)|\mathbf{1}_{\{X^{t,\beta}(\tau_0(t))\le\varepsilon, |X^x(\tau_0(t)))-x|>\varepsilon'\}}\right] \\
&\le 2\|\phi\|_\infty \mathbf{P}[X^{t,\beta}(\tau_0(t)) > \varepsilon] + \delta + 2\|\phi\|_\infty \mathbf{P}[|X^x(\tau_0(t))) - x| > \varepsilon']
\end{aligned}$$

Now, by Proposition 4.4.1, for all $t \le \delta^{\frac{1}{\beta}}\varepsilon$ it holds that $\mathbf{P}[X^{t,\beta}(\tau_0(t)) > \varepsilon] \le \delta$. Then the estimate above reads

$$|\tilde{u}(t,x) - \phi(0,x)| \le 2\|\phi\|_\infty\delta + \delta + 2\|\phi\|_\infty \mathbf{P}[|X^x(\tau_0(t))) - x| > \varepsilon'], \quad \text{for every } t \le \delta^{\frac{1}{\beta}}\varepsilon.$$

To conclude, by stochastic continuity, pick a possibly smaller threshold $\bar{t}$ to obtain

$$\mathbf{P}[|X^x(\tau_0(t))) - x| > \varepsilon'] \le \delta \quad \text{for every } t \le \bar{t}.$$

□

Remark 4.4.3. The continuity at $t = 0$ of Proposition 4.4.2 is not obvious. For example it is clear that Proposition 4.4.2 fails if we replace $-X^{t,\beta}$ with a decreasing Poisson process. In fact Proposition 4.4.2 fails in general if we replace $-X^{t,\beta}$ with a decreasing compound Poisson process $-N^t(s)$ with generator

$$-D_\infty^{(\nu)} f(t) := \int_0^\infty (f(t-r) - f(t))\,\nu(dr), \quad \text{where} \quad 0 < \lambda := \int_0^\infty \nu(dr) < \infty.$$

To see this, observe that for every $\varepsilon, t > 0$

$$\mathbf{P}\left[N^t\left(\tau_0(t)\right) > \varepsilon\right] \geq \mathbf{P}\left[\text{first jump of } N^t \text{ is greater than } t+\varepsilon\right] = \int_{t+\varepsilon}^{\infty} \frac{\nu(dr)}{\lambda},$$

and note that the right hand side is non-decreasing as $t \downarrow 0$, where τ_0 is the left continuous inverse of N^0. As $\int_0^\infty \nu(dr) > 0$ we can choose $\varepsilon_0 > 0$ and $\bar{t} > 0$ so that

$$\inf_{t \leq \bar{t}} \mathbf{P}\left[N^t\left(\tau_0(t)\right) > \varepsilon_0\right] \geq \int_{\bar{t}+\varepsilon_0}^{\infty} \frac{\nu(dr)}{\lambda} =: c > 0.$$

Now, consider a continuous non-negative ϕ with $\phi(0) = 0$, such that $\inf_{r \in (-\infty, -\varepsilon_0]} \phi(r) > 0$. Then for every $t \leq \bar{t}$

$$\begin{aligned}
|\tilde{u}(t) - \phi(0)| &= \mathbf{E}\left[\phi(-N^t\left(\tau_0(t)\right)\left(\mathbf{1}_{\{N^t(\tau_0(t))>\varepsilon_0\}} + \mathbf{1}_{\{N^t(\tau_0(t))\leq\varepsilon_0\}}\right)\right] \\
&\geq \mathbf{E}\left[\phi(-N^t\left(\tau_0(t)\right))\mathbf{1}_{\{N^t(\tau_0(t))>\varepsilon_0\}}\right] \\
&\geq \inf_{r \in (-\infty, -\varepsilon_0]} \phi(r) \mathbf{P}\left[N^t\left(\tau_0(t)\right) > \varepsilon_0\right] \\
&\geq \inf_{r \in (-\infty, -\varepsilon_0]} \phi(r) c > 0.
\end{aligned}$$

Chapter 5

Numerical results and intuition

5.1 Numerical results

In this section, we present some numerical results to verify the stochastic representation formula. To this end, we consider the one-dimensional nonlocal diffusion problem (4.1) in the unit interval $\Omega = (-1, 1)$. We use the notation of Chapter 3.

5.1.1 Non-integrable kernels

We start with the case of non-integrable kernel function

$$\nu_\delta(r) = (1-\alpha)\delta^{\alpha-1} r^{-\alpha-1} \mathbf{1}_{(0,\delta)}(r), \tag{5.1}$$

with $\alpha \in (0, 1)$ and the following data.

(a) initial data $\phi(x, t) = e^{5t}(1 + x)(1 - x)^2 x$ and zero source term $f \equiv 0$;

(b) trivial initial data $\phi(x, t) = 0$ and source term $f = \sin(10t)(1 - x)x \sin(\pi x)$.

The kernel function is proposed in this way in order to keep that $\int_0^\delta r\nu_\delta(r)\, dr = 1$ and hence the nonlocal operator recovers the infinitesimal first-order derivative as the nonlocal horizon diminishes. The analytical property of the model has been extensively studied in Du et al. [2017].

The stochastic process generated by spatially second-order derivative (with zero boundary conditions), which is well-known as the killed Brownian motion in the domain $\Omega = (-1, 1)$, can be simply approximated by the lattice random walk. Specifically, we divided the interval $(-1, 1)$ into M small intervals, with the uniform mesh size $h = 2/M$ and grid points $x_j = jh - 1$, $j = 0, 1, 2, \ldots, M$. Then in each time level, the particle standing in the grid points x_j will randomly move to grid points x_{j-1} or x_{j+1}. In case that the particle hits the boundary of Ω, then the time is set as $\tau_\Omega(x_j)$. Here we let $B_h^{x_j}(t)$ be the position where the particle starting at position x_j arrives at time t.

Similarly, the stochastic process generated by the operator

$$-D_\delta^{(\nu)} u(t) = -\int_0^\delta (u(t) - u(t-r))\nu_\delta(r)\, dr$$

with historical initial data could also be approximated by a one-dimensional lattice random walk, where the trajectory of the particle involves some long-distance jumps. To numerically simulate the stochastic process, we discretize $[0, T]$ into N small intervals $[t_{n-1}, t_n]$ with $n = 1, 2, \dots, N$ and let $k = T/N$. Then we consider the discretization (assume that $\delta = mk$)

$$
\begin{aligned}
D_\delta^{(\nu)} u(t_n) =& \int_0^k (u(t_n) - u(t_n - r))\nu_\delta(r)\, dr + \sum_{j=2}^m \int_{(j-1)k}^{jk} (u(t_n) - u(t_n - r))\nu_\delta(r)\, dr \\
\approx& \frac{u(t_n) - u(t_{n-1})}{k} \int_0^k r\nu_\delta(r)\, ds + \sum_{j=2}^m (u(t_n) - u(t_{n-k})) \int_{(j-1)k}^{jk} \nu_\delta(r)\, dr \\
=& \frac{1}{k^\alpha}\Big(\omega_0 u(t_n) - \sum_{j=1}^m \omega_j u(t_{n-j})\Big) =: \bar{D}_\delta^{(\nu)} u(t_n).
\end{aligned}
\tag{5.2}
$$

Here the weights $\{\omega_j\}_{j=0}^m$ are computed exactly as

$$
\omega_0 = \delta^{\alpha-1}\Big(1 + \frac{1-\alpha}{\alpha}(1 - m^{-\alpha})\Big), \quad \omega_1 = \delta^{\alpha-1}
$$

and

$$
\omega_j = \delta^{\alpha-1}\frac{1-\alpha}{\alpha}((j-1)^{-\alpha} - j^{-\alpha}), \quad j = 2, 3, \dots, m.
$$

At each time level, the particle standing at the grid point t_j will jump to one of the grid points t_{j-i}, for $i = 1, 2, ..., m$, with the probability $p_i = \omega_i/\omega_0$. It is easy to verify that $\sum_{j=1}^m \omega_j = \omega_0$ and hence $\sum_{j=1}^m p_j = 1$. We let $\tau_0(t_n)$ be the time that the particle starting at t_n passes 0, and $X_k^{t_n,(\nu)}(\tau_0(t_n))$ be the position where the particle arrives below 0. Then by applying the scaling $2\alpha k^\alpha = h^2\delta^{\alpha-1}$, the solution of the nonlocal-in-time evolution equation (4.1) can be computed by

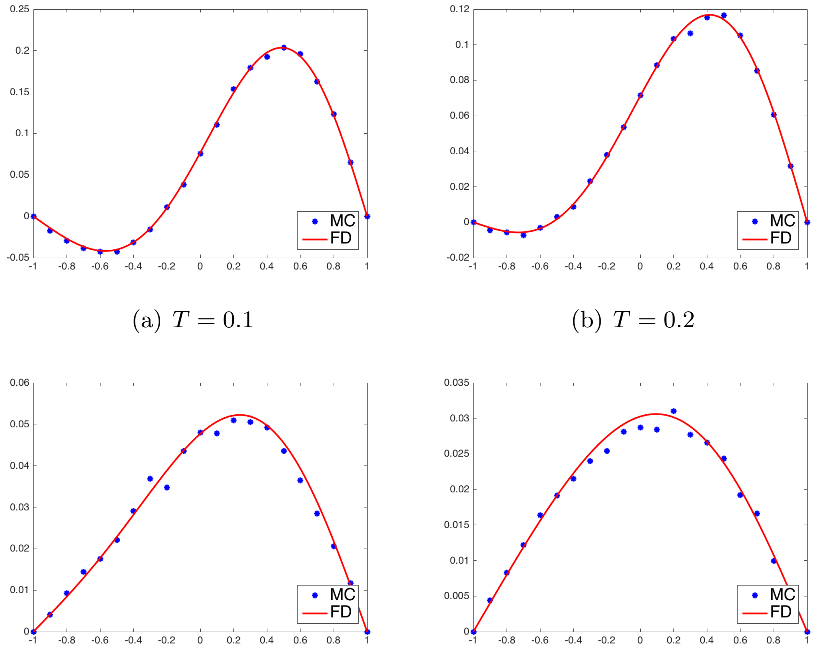
$$
\begin{aligned}
U_h^n =& \mathbf{E}\left[\phi\left(-X_k^{t_n,(\nu)}(\tau_0(t)), B_h^{x_j}(\tau_0(t_n))\right)\mathbf{1}_{\{\tau_0(t_n)<\tau_\Omega(x_j)\}}\right] \\
&+ \mathbf{E}\left[\int_0^{\tau_0(t_n)\wedge\tau_\Omega(x_j)} f\left(-X_k^{t_n,(\nu)}(s), B_h^{x_j}(s)\right) ds\right],
\end{aligned}
$$

using the Monte Carlo method, where the integral of the second term is computed by the trapezoid rule.

In Figures 5.1 and 5.2, we plot the numerical solution of nonlocal-in-time diffusion model (4.1) where the nonlocal operator involves the finite-horizon kernel function (5.1) with $\alpha = 0.75$ and $\delta = 0.2$, at different time levels, $T = 0.1$, 0.2, 0.4 and 0.6. To compute the numerical solution, we let $h = 0.02$ and $k = \sqrt[\alpha]{h^2\delta^{\alpha-1}/2\alpha}$, and use 50000 Monte Carlo trials. Since the closed form of the analytical solution is not available, the benchmark solutions are computed by finite difference scheme

$$
\bar{D}_\delta^{(\nu)} u_h^n - \bar{\partial}_{xx}^h u_h^n = f^n
$$

with a very fine mesh, say $k = 10^{-4}$ and $h = 10^{-3}$, where the discrete operator in time $\bar{D}_\delta^{(\nu)}$ is given by (5.2) and the spatial one $\bar{\partial}_{xx}^h$ is the central difference approximation to the second order



(a) $T = 0.1$ (b) $T = 0.2$

(c) $T = 0.4$ (d) $T = 0.6$

Figure 5.1: Numerical solutions of Example (a) with $\delta = 0.2$ and $\alpha = 0.75$. (Blue dots: numerical solutions computed by the stochastic representation and Monte Carlo method (MC), with $h = 0.02$, $k = \sqrt[\alpha]{h^2\delta^{\alpha-1}/2\alpha}$ and 50000. Red curves: reference solutions computed by finite difference method (FD) with $h = 10^{-3}$ and $k = 10^{-4}$.)

derivative. In Figures 5.1 and 5.2, the solution computed using the stochastic representation formula and the Monte Carlo method (MC) is plotted by blue dots while the finite difference solution (FD) is plotted by the red curves. We observe that the numerical solution computed by the stochastic approach is very close to the one computed by the finite difference scheme, which supports our theoretical results.

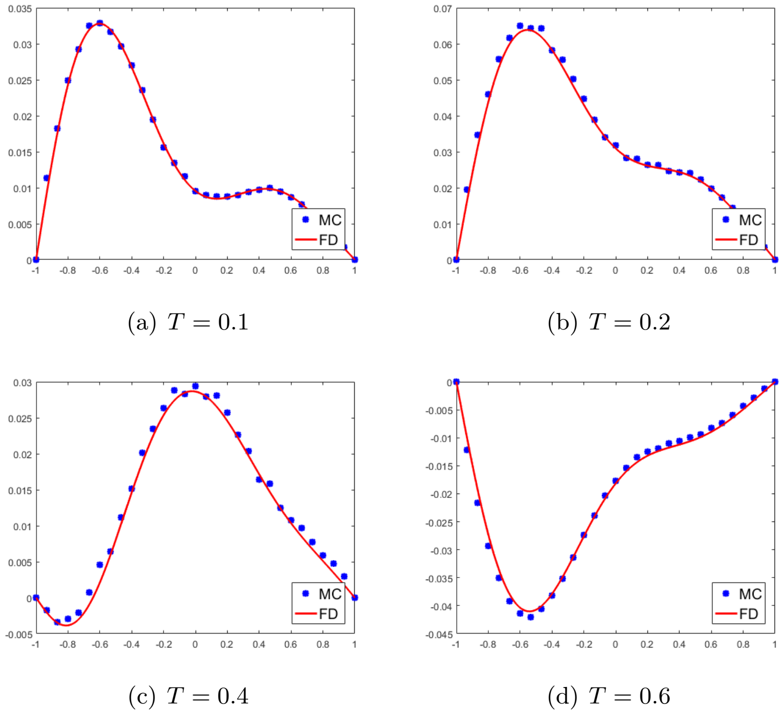
5.1.2 Integrable kernels

Next, we present some numerical results for a special integrable kernel which is the Dirac measure concentrated at $\delta > 0$ weighted by $\lambda > 0$, i.e.,

$$-D_{\infty}^{(\nu)} u(t) := (u(t-\delta) - u(t))\lambda.$$

This nonlocal operator is the generator of a decreasing Poisson process, which performs negative jumps of size δ after a λ-exponential waiting time. Hence we have

$$t - X^{(\nu)}(\tau_0(t)) = t - n\delta \quad \text{a.s.,} \quad \text{for} \quad t \in ((n-1)\delta, n\delta], \quad n \in \mathbb{N},$$



(a) $T = 0.1$ (b) $T = 0.2$

(c) $T = 0.4$ (d) $T = 0.6$

Figure 5.2: Numerical solutions of Example (b) with $\delta = 0.2$ and $\alpha = 0.75$. (Blue dots: numerical solutions computed by the stochastic representation and Monte Carlo method (MC), with $h = 0.02$, $k = \sqrt[\alpha]{h^2\delta^{\alpha-1}/2\alpha}$ and 50000 trials. Red curves: reference solutions computed by finite difference method (FD) with $h = 10^{-3}$ and $k = 10^{-4}$.)

and $\tau_0(t)$ is a Gamma(n, λ) random variable. Then solution to problem (4.1) with zero source term $f = 0$ allows the stochastic representation (3.9) for $t \in ((n-1)\delta, n\delta]$, $n \in \mathbb{N}$,

$$u(t, x) = \mathbf{E}\left[\phi\left(t - n\delta, B^x(\tau_0(t))\right) \mathbf{1}_{\{\tau_0(t) < \tau_\Omega(x)\}}\right].$$

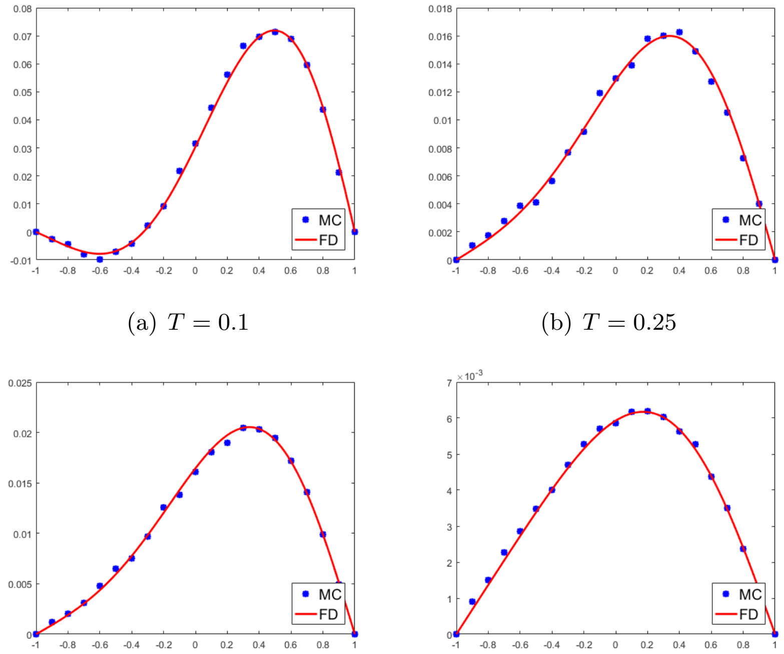
Note that even if $\phi \in C^\infty([-\delta, 0] \times \overline{\Omega})$, in general

$$\lim_{t \downarrow 0} u(t, x) = \mathbf{E}[\phi(-\delta, B^x(\tau_0(1)))\mathbf{1}_{\{\tau_0(1) < \tau_\Omega(x)\}}] \neq \phi(0, x).$$

In Figure 5.1.2, we plot the numerical solutions (blue dots) with $\lambda = 1$ at different time levels, where $h = 0.04$ and 50000 Monte Carlo trials are used. Again, the reference solutions, plotted by red curves, are computed by the finite difference method

$$\lambda(u_h^n - u_h^{n-\delta/k}) - \bar{\partial}_{xx} u_h^n = f^n, \quad n = 1, 2, \ldots, N,$$

with very fine meshes, i.e., $k = 10^{-3}$ and $h = 10^{-3}$. Numerical results show that the Monte Carlo simulation using the Feynman-Kac formula approximates the solution very well.



(a) $T = 0.1$ (b) $T = 0.25$

(c) $T = 0.3$ (d) $T = 0.45$

Figure 5.3: Numerical solutions for the integrable kernel with $\lambda = 1$ and $\delta = 0.2$. (Blue dots: numerical solutions computed by MC, with $h = 0.04$ and 50000 trials. Red curves: reference solutions computed by FD with $h = 10^{-3}$ and $k = 10^{-3}$.)

5.2 Intuition for the time-nonlocal initial conditions

We discuss the intuition for the stochastic representation (4.3) as the solution to the Marhcaud EE (4.1) studied in Chapter 4. Let us write $-W(t) = t - X^\beta(\tau_0(t)) = -X^{t,\beta}(\tau_0(t))$. Then $W(t)$ is the overshoot of the subordinator X^β with respect to the barrier t, recalling that the first exit time/inverse subordinator is given by $\tau_0(t) = \inf\{s > 0 : \ t \le X^\beta(s)\}$. To ease notation we write $Y^x := \{X^{x,\alpha}(\tau_0(t))\mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}}\}_{t\ge 0}$. Let us start from the intuition of Caputo EEs, as if $\phi(t,x) = \phi(0,x) =: \phi_0(x)$ for every $t \in (-\infty, 0] \times \Omega$, then the solution (4.3) reads

$$u(t,x) = \mathbf{E}\left[\phi_0(Y^x(t))\right], \tag{5.3}$$

and the EE (4.1) equals the Caputo EE (4.6) (for $g = f = 0$). The probabilistic object defining the solution (5.3) is the anomalous diffusion Y^x. Recall that the particle Y^x is either trapped or diffusing.

Key observation: reasoning path-wise, for some $\bar{x} \in \Omega$

the interval (t_1, t_2) is the maximal open interval so that $t \mapsto Y^x(t) = \bar{x}$ is constant

$$\Longleftrightarrow$$

the interval (t_1, t_2) is the maximal open interval so that $t \mapsto \tau_0(t)$ is constant

$$\Longleftrightarrow$$

$$\text{the interval } (t_1,t_2) \text{ is the maximal open interval so that } t \mapsto X^\beta(\tau_0(t)) \text{ is constant}$$
$$\Longleftrightarrow$$
$$X^\beta(\tau_0(t)-) = t_1 \text{ and } X^\beta(\tau_0(t)) = t_2, \text{ (i.e. } X^\beta \text{ jumped from } t_1 \text{ to } t_2).$$

The last statement implies that

$$W(t) = X^\beta(\tau_0(t)) - t = t_2 - t \in (0, t_2 - t_1) \text{ for every } t \in (t_1, t_2),$$

which is the trapping/waiting time of $Y^x(t)$. In words: the event of the diffusion Y^x being trapped at a point $\bar{x} \in \Omega$ at time t until time $t+s$ happens precisely when $W(t) = s$. Hence the law of $-W(t)$ provides a weighting of the initial condition $\phi(\bar{x})$ depending on the trapping/waiting time of $Y^x(t)$. Notice that the process $t \mapsto -W(t)$ is self-similar with index 1 and it is composed by right continuous 45 degrees increasing slopes with 0 leftmost limit (see Figure 1).

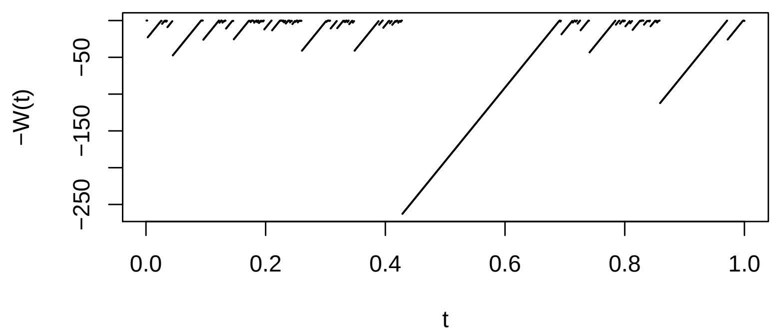


Figure 5.4: A typical path of the overshoot $t \mapsto -W(t) = -X^{t,\beta}(\tau_0(t))$, $\beta = 0.9$.

A non-memory interpretation

It is possibly appealing to think about the values $(-\infty, 0) \times \Omega$ for the initial condition ϕ as the 'depth' underneath the surface $\{0\} \times \Omega$ where the particle Y^x moves. Then one can think about the particle $Y^x(t)$ as falling instantaneously at the bottom of a hole/trap of depth $|t_2 - t_1|$, and then taking time $|t_2 - t_1|$ to climb back up to the surface. Then, at time t one can observe the particle being $|t_2 - t|$-depth-units down in the hole. From this viewpoint, once the particle is in the hole it just drifts upward with unit speed. As a quick example, consider the variable separable initial condition $\phi(t,x) = p(t)q(x)$ where $p(t) = \mathbf{1}_{\{t<-1\}}$. Then the solution reads for

$t > 0$

$$\begin{aligned}
u(t,x) &= \mathbf{E}\left[q(Y^x(t))\mathbf{1}_{\{W(t)>1\}}\right] \\
&= \mathbf{E}\left[q(Y^x(t))|Y^x(t) \text{ is more than 1 unit deep in a trap}\right] \\
\Big(&= \mathbf{E}\left[q(Y^x(t))|Y^x(t) \text{ is trapped for more than 1 time-unit}\right]\Big).
\end{aligned}$$

Hence, in this example the diffusive particle Y^x will have to be a least a unit deep in a hole (trapped for at least a unit time) for the values at its trapping point at its depth (in the past) to contribute to the solution.

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